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Applications of the Clifford algebra valued boundary element method to electromagnetic scattering problems

Jia-Wei Lee [a], Li-Wei Liu [b], Hong-Ki Hong [b], Jeng-Tzong Chen [a,c,*]

[a] Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan
[b] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan
[c] Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung, Taiwan

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ABSTRACT

Electromagnetic problems governed by Maxwell's equations are solved by using a Clifford algebra valued boundary element method (BEM). The well-known Maxwell's equations consist of eight pieces of scalar partial differential equations of the first order. They can be rewritten in terms of the language of Clifford analysis as a nonhomogeneous *k*-Dirac equation with a Clifford algebra valued function. It includes three-component electric fields and three-component magnetic fields. Furthermore, we derive Clifford algebra valued boundary integral equations (BIEs) with Cauchy-type kernels and then develop a Clifford algebra valued BEM to solve electromagnetic scattering problems. To deal with the problem of the Cauchy principal value, we use a simple Clifford algebra valued *k*-monogenic function to exactly evaluate the Cauchy principal value. Free of calculating the solid angle for the boundary point is gained. The remaining boundary integral is easily calculated by using a numerical quadrature except the part of Cauchy principal value. This idea can also preserve the flexibility of numerical method, hence it is suitable for any geometry shape. In the numerical implementation, we introduce an oriented surface element instead of the unit outward normal vector and the ordinary surface element. In addition, we adopt the Dirac matrices to express the bases of Clifford algebra $Cl_3(\mathbb{C})$. We also use an orthogonal matrix to transform global boundary densities into local boundary densities for satisfying boundary condition straightforward. Finally, two electromagnetic scattering problems with a perfect spherical conductor and a prolate spheroidal conductor are both considered to examine the validity of the Clifford algebra valued BEM with Cauchy-type kernels.

1. Introduction

Complex algebra and complex analysis are powerful tools for solving problems in two-dimensional spaces. This point motivates scholars to develop methods based on complex algebra/analysis to deal with problems in three-dimensional spaces. However, due to the algebraic structures of complex numbers and the calculus of complex variables, complex algebra/analysis are improper to be the foundation to naturally develop any numerical method for solving problems in three-dimensional spaces. To preserve some benefits of complex algebra or complex analysis for solving three-dimensional problems, an extension of complex algebra is required.

In 1843, Hamilton [1] proposed quaternion algebra which is an extension of complex algebra in three-dimensional space. Therefore, complex algebra can be seen as a subalgebra of quaternion algebra. Although the multiplication of two quaternion numbers is non-commutative, quaternion algebra can be applied to describe three-dimensional problems. For quaternionic analysis, Fueter and his collaborators started developing it since 1930 [2–5].

On the other hand, Clifford [6] proposed the algebras named after him in 1878. Clifford algebra can be seen as an extension of complex or quaternionic algebras. Bases of Clifford algebra are generated according to the Clifford product rule. In this way, Clifford algebra is different from other algebraic systems that it has no more and no fewer bases to describe the geometric relations of space [7,8] and easier to extend it to higher dimensional problems. As quoted by *Hestenes*: 「*Geometry without algebra is dumb! - Algebra without geometry is blind!*」 [8]. A linear combination of those bases is called Clifford number or multivector. Later, Hestenes [7,9] considered that Clifford algebra can be a common language in physics and mathematics. Clifford algebra has been applied to many fields such as, geometry, dynamics, physics, electromagnetics and information theory [10,11].

As well as the complex analysis in the complex variables and quaternionic analysis [12–14] in the quaternion algebra, there is a new field, Clifford analysis or so-called hypercomplex analysis [15–

* Corresponding author at: Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan.
E-mail addresses: 29952008@mail.ntou.edu.tw, handsome6361@gmail.com (J.-W. Lee), liweiliu@ntu.edu.tw (L.-W. Liu), hkhong@ntu.edu.tw (H.-K. Hong), jtchen@mail.ntou.edu.tw (J.-T. Chen).

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20] in Clifford algebra. Liu and Hong used Clifford analysis to derive general solutions of both isotropic elasticity [21] and anisotropic elasticity [22,23]. One goal of this paper is to develop the Clifford algebra valued BIE with the Cauchy-type kernel to deal with three-dimensional problems. Regarding Clifford algebra valued BIE [24], Hong and Liu derived it [25,26] and employed it to solve three-dimensional magnetostatic problems [27] and three-dimensional elasticity [28]. However, they merely focused on static problems.

Now, we extend Clifford algebra valued BIE to solve three-dimensional time-harmonic problems. For time-harmonic problems, such as Helmholtz equations, Gerus and Shapiro [29] derived a Cauchy-type integral corresponding to the two-dimensional Helmholtz equation by using quaternion algebra. Later, Vu Thi Ngoc Ha and Begehr [30] extended to the three-dimensional Helmholtz equation. Both works focused on deriving the Cauchy-type integral, however no numerical results were provided. Chantaveerod and his coworkers [31–33] employed the four-dimensional Clifford algebra to describe Maxwell's equations as a k-Dirac equation. They called it Maxwell-Dirac Equation. They also employed the corresponding Clifford algebra valued BIE to calculate both interior and exterior problems with a plane wave and a Hertzian dipole, respectively.

In this paper, we employ the three-dimensional Clifford algebra $Cl_3(\mathbb{C})$ and Clifford analysis to reformulate Maxwell's equations to a k-Dirac equation. We believe that algebraic space of the three-dimensional Clifford algebra is sufficient to describe Maxwell's equations. Furthermore, we derive Clifford algebra valued BIEs for the k-Dirac equation and develop its Clifford algebra valued BEM. To examine the validity of Clifford algebra valued BEM with the Cauchy-type kernel, two electromagnetic scattering problem with a perfect conductor and a prolate spheroidal conductor are considered. Finally, the numerical results obtained from Clifford algebra valued BEM show a good agreement with those of finite element method (FEM) [34] and method of fundamental solutions (MFS) [35].

2. Problem statement of electromagnetic scattering

The typical electromagnetic scattering problem in the frequency domain is governed by Maxwell's equations as shown below:

$$\nabla\cdot\vec{D}(x) = \rho_f(x), \tag{1}$$

$$\nabla\times\vec{H}(x) - i\omega\vec{D}(x) = \vec{J}_f(x), \tag{2}$$

$$\nabla\times\vec{E}(x) + i\omega\vec{B}(x) = 0, \tag{3}$$

$$\nabla\cdot\vec{B}(x) = 0, \tag{4}$$

where $\vec{E}(x)$ is the electric field, $\vec{D}(x)$ is the electric displacement field, $\vec{B}(x)$ is the magnetic flux density, $\vec{H}(x)$ is the magnetic field strength, $\vec{J}_f(x)$ is the free current density, $\rho_f(x)$ is the free charge density and ω is the angular frequency. For the linear isotropic medium, the constitutive equations are shown below:

$$\vec{D}(x) = \varepsilon\vec{E}(x), \tag{5}$$

$$\vec{H}(x) = \frac{1}{\mu}\vec{B}(x), \tag{6}$$

where ε and μ are the permittivity and the permeability, respectively. The relation between ε and μ is

$$\frac{1}{\sqrt{\varepsilon\,\mu}} = \frac{\omega}{k} = c, \tag{7}$$

where k and c denote the wave number and the speed of the electromagnetic wave. In a vacuum, it is usually to use the symbols ε_0 and μ_0 to stand for the permittivity and the permeability, respectively. The values of ε_0 and μ_0 are

$$\varepsilon_0 \approx 8.854187817\times10^{-12}\ \mathrm{A^2{\cdot}s^4{\cdot}kg^{-1}{\cdot}m^{-3}}, \tag{8}$$

$$\mu_0 = 4\pi\times10^{-7}\ \mathrm{A{\cdot}s^{-2}{\cdot}kg{\cdot}}m, \tag{9}$$

respectively. Also, c_0 is the speed of light in a vacuum and its value is

$$c_0 = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 299792458\ \mathrm{m{\cdot}s^{-1}}. \tag{10}$$

Substituting Eqs. (5) to (7) into Eqs. (1) to (4), we obtain

$$\nabla\cdot\vec{E}(x) = \frac{\rho_f(x)}{\varepsilon}, \tag{11}$$

$$\nabla\times\frac{\vec{B}(x)}{\sqrt{\varepsilon\,\mu}} - ik\vec{E}(x) = \sqrt{\frac{\mu}{\varepsilon}}\,\vec{J}_f(x), \tag{12}$$

$$\nabla\times\vec{E}(x) + ik\frac{\vec{B}(x)}{\sqrt{\varepsilon\,\mu}} = 0, \tag{13}$$

$$\nabla\cdot\vec{B}(x) = 0. \tag{14}$$

The boundary conditions for an electromagnetic scattering problem with a perfect conductor are

$$\vec{n}(x)\times\vec{E}(x) = 0, \tag{15}$$

$$\vec{n}(x)\cdot\vec{B}(x) = 0, \tag{16}$$

where $\vec{n}(x)$ is a unit inward normal vector on the conductor surface and

$$\vec{E}(x) = \vec{E}^{in}(x) + \vec{E}^{s}(x),\quad \vec{E}^{(\cdot)}(x) = (E_1^{(\cdot)}(x), E_2^{(\cdot)}(x), E_3^{(\cdot)}(x)), \tag{17}$$

$$\vec{B}(x) = \vec{B}^{in}(x) + \vec{B}^{s}(x),\quad \vec{B}^{(\cdot)}(x) = (B_1^{(\cdot)}(x), B_2^{(\cdot)}(x), B_3^{(\cdot)}(x)), \tag{18}$$

in which the superscripts "*in*" and "*s*" stand for the incident wave and the scattering field, respectively.

3. Clifford algebra and Clifford analysis in $Cl_3(\mathbb{C})$

3.1. Algebraic structures of Clifford algebra $Cl_3(\mathbb{C})$

The Clifford product e_je_k (in that order, denoted by juxtaposition) of e_j and e_k is defined by the Clifford product rule

$$e_je_k + e_ke_j = 2\delta_{jk},\ j,\ k = 1,\ 2,\ 3. \tag{19}$$

The basis elements of Clifford algebra in three-dimensional Euclidean space are generated from $e_\varnothing = 1$ and $\{e_j\}$ by the rule of Eq. (19) and we have eight bases

$$1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123}, \tag{20}$$

where $e_{23} = e_2e_3$, $e_{31} = e_3e_1$, $e_{12} = e_1e_2$ and $e_{123} = e_1e_2e_3$. For the complex-valued Clifford algebra $Cl_3(\mathbb{C})$, a Clifford number or a so-called multivector is constructed by Eq. (20) as shown below,

$$a = a_0 1 + a_1e_1 + a_2e_2 + a_3e_3 + a_{23}e_{23} + a_{31}e_{31} + a_{12}e_{12} + a_{123}e_{123}, \tag{21}$$

where

$$a_0, a_1, a_2, a_3, a_{23}, a_{31}, a_{12}, a_{123}, \in \mathbb{C} \tag{22}$$

For a Clifford algebra valued function in $Cl_3(\mathbb{C})$, we can define a function in the Euclidean space $\mathbb{R}^3$ as

$$f(x): \mathbb{R}^3 \to Cl_3(\mathbb{C}), \tag{23}$$

where

$$\begin{aligned} f(x) &= f_0(x)1 + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3 \\ &\quad + f_{23}(x)e_{23} + f_{31}(x)e_{31} + f_{12}(x)e_{12} + f_{123}(x)e_{123}, \end{aligned} \tag{24}$$

in which $f_{(\cdot)}(x)$ is a complex-valued function, the domain and range of Clifford algebra valued function $f(x)$ are $\mathbb{R}^3$ and $Cl_3(\mathbb{C})$, respectively. For more information on the basic acknowledge of Clifford algebra and Clifford analysis, readers may consult the edited book by Ablamowicz and Sobczyk book [10].

3.2. Dirac-type operator and Cauchy-type kernel in $Cl_3(\mathbb{C})$

Using the language of Clifford algebra, the Helmholtz operator can be expressed as

$$\Delta + k^2 = \nabla^2 + k^2 = -\bar{D}_k D_k, \tag{25}$$

where $k \in \mathbb{C}$ is the wave number and the D_k is the k-Dirac operator and is defined as

$$D_k = D_x + ik = \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3 + ik, \tag{26}$$

and its Clifford conjugate is

$$\bar{D}_k = -D_x + ik = -\frac{\partial}{\partial x_1}e_1 - \frac{\partial}{\partial x_2}e_2 - \frac{\partial}{\partial x_3}e_3 + ik, \tag{27}$$

where D_x is the Dirac operator. If a Clifford algebra valued function $f(x)$ satisfies the k-Dirac equation

$$D_k f(x) = 0, \tag{28}$$

$f(x)$ is called a Clifford algebra valued k-monogenic function. The fundamental solution $U(x, s)$ of the Helmholtz equation and the Cauchy-type kernel $C_k(x, s)$ satisfy

$$-\left(\Delta + k^2\right)U(x, s) = \bar{D}_k D_k U(x, s) = U(x, s)D_k\bar{D}_k = \delta(x - s), \tag{29}$$

and

$$\bar{D}_k C(x, s) = C(x, s)\bar{D}_k = \delta(x - s), \tag{30}$$

respectively. Then, we have the fundamental solution

$$U(x, s) = \frac{1}{4\pi}\frac{e^{-ik|x-s|}}{|x - s|}, \tag{31}$$

and the Cauchy-type kernel

$$\begin{aligned} C_k(x, s) &= D_k U(x, s) = U(x, s)\bar{D}_k \\ &= \frac{1}{4\pi}\left[\left(\frac{-ik}{|x - s|} - \frac{1}{|x - s|^2}\right)\frac{e^{-ik|x-s|}}{|x - s|}(x - s) + ik\frac{e^{-ik|x-s|}}{|x - s|}\right], \end{aligned} \tag{32}$$

where $x - s = (x_1 - s_1)e_1 + (x_2 - s_2)e_2 + (x_3 - s_3)e_3$

4. Clifford analysis and Clifford algebra valued BIE for electromagnetic scattering problems

4.1. Clifford algebra valued Maxwell's equation

According to a Clifford algebra valued function, we have

$$\begin{aligned} f(x) &= E(x) + c\,B(x) \\ &= E_1(x)e_1 + E_2(x)e_2 + E_3(x)e_3 + c\big(B_1(x)e_{23} \\ &\quad + B_2(x)e_{31} + B_3(x)e_{12}\big), \end{aligned} \tag{33}$$

Based on the language of k-Dirac operator D_k, Maxwell's equations in Eqs. (11)–(14) is nothing but a nonhomogeneous k-Dirac equation [10,11]

$$D_k f(x) = p(x), \tag{34}$$

where $p(x)$ is

$$p(x) = \frac{\rho_f(x)}{\varepsilon} - \sqrt{\frac{\mu}{\varepsilon}}\left(\vec{J}_{f1}(x)e_1 + \vec{J}_{f2}(x)e_2 + \vec{J}_{f3}(x)e_3\right). \tag{35}$$

Note that Eq. (34) contains eight pieces of scalar partial differential equations of the first order in Maxewll's equations. The nonhomogeneous k-Dirac equation in Eq. (35) is equivalent to Maxewll's equations (Eqs. (11)–(14)).

4.2. Clifford algebra valued boundary integral equation for nonhomogeneous k-Dirac equation

In Clifford analysis, Stokes' theorem can be represented by

$$\begin{aligned} &\int_{\partial\Omega} g(x)n(x)h(x)dS(x) \\ &\quad = \int_{\Omega}\left[\left(g(x)D_x\right)h(x) + g(x)\left(D_x h(x)\right)\right]d^3x, \end{aligned} \tag{36}$$

where $n(x)$ is the Clifford algebra valued unit outward normal vector of the field point x on the boundary $\partial\Omega$, $g(x)$ and $h(x)$ are Clifford algebra valued functions. For the Helmholtz and the k-Dirac equations, we derive the corresponding Clifford algebra valued BIEs from Stokes' theorem. By substituting $g(x) = \Psi(x)D_x$ and $h(x) = \Phi(x)$ into Eq. (36), we have

$$\begin{aligned} \int_{\partial\Omega}\left(\Psi(x)D_x\right)n(x)\Phi(x)dS(x) &= \int_{\Omega}\left(\Psi(x)D_xD_x\right)\Phi(x)\,d^3x \\ &\quad + \int_{\Omega}\left(\Psi(x)D_x\right)\left(D_x\Phi(x)\right)d^3x. \end{aligned} \tag{37}$$

Similarly, substituting $g(x) = \Psi(x)$ and $h(x) = D_x\Phi(x)$ into Eq. (36), we have

$$\begin{aligned} \int_{\partial\Omega}\Psi(x)n(x)\left(D_x\Phi(x)\right)dS(x) &= \int_{\Omega}\left(\Psi(x)D_x\right)\left(D_x\Phi(x)\right)d^3x \\ &\quad + \int_{\Omega}\Psi(x)\left(D_xD_x\Phi(x)\right)d^3x. \end{aligned} \tag{38}$$

By subtracting Eq. (38) from Eq. (37), we have

$$\begin{aligned} &\int_{\Omega} -\left(\Psi(x)\bar{D}_kD_k\right)\Phi(x)\,d^3x + \int_{\Omega}\Psi(x)\left(\bar{D}_kD_k\Phi(x)\right)d^3x \\ &= \int_{\partial\Omega}\left(\Psi(x)D_k\right)n(x)\Phi(x)dS(x) - \int_{\partial\Omega}\Psi(x)n(x)\left(D_k\Phi(x)\right)dS(x). \end{aligned} \tag{39}$$

We term Eq. (39) the Clifford algebra valued Green's third identity. Similarly, substituting $\Phi(x) = f(x)$ and $\Psi(x) = U(x, s)$ into Eq. (39), we have

$$\int_{\Omega} \delta(x-s) f(x) d^3x = -\int_{\partial\Omega} \left(U(x,s)D_k\right) n(x) f(x) dS(x) + \int_{\partial\Omega} U(x,s) n(x) \left(D_k f(x)\right) dS(x) + \int_{\Omega} U(x,s)\left(\bar{D}_k p(x)\right) d^3x. \tag{40}$$

where $U(x,s)$ satisfies Eq. (29) and $f(x)$ satisfies the non-homogeneous Helmholtz equation. Similarly, according to the location of the source point s, we have the Clifford algebra valued BIEs as shown below

$$c(s) f(s) = -\int_{\partial\Omega}^{(s)} C_k(x,s) n(x) f(x) dS(x) + \int_{\partial\Omega} U(x,s) n(x)\left(D_k f(x)\right) dS(x) + \int_{\Omega} U(x,s)\left(\bar{D}_k p(x)\right) d^3x, \tag{41}$$

where

$$c(s) = \begin{cases} 1, & s \in \Omega, \\ \dfrac{\alpha}{4\pi}, & s \in \partial\Omega, \\ 0, & s \in \mathbb{R}^3/\mathrm{cl}\Omega, \end{cases} \tag{42}$$

and

$$\int_{\partial\Omega}^{(s)} = \begin{cases} \int_{\partial\Omega}, & s \in \Omega, \\ C.P.V. \int_{\partial\Omega}, & s \in \partial\Omega, \\ \int_{\partial\Omega}, & s \in \mathbb{R}^3/\mathrm{cl}\Omega. \end{cases} \tag{43}$$

If the Clifford algebra valued function $f(x)$ is just a k-monogenic function, the Clifford algebra valued BIEs can be reduced to

$$c(s) f(s) = -\int_{\partial\Omega}^{(s)} C_k(x,s) n(x) f(x) dS(x) + P(s), \tag{44}$$

where $P(s)$ is

$$P(s) = \int_{\Omega} C_k(x,s) p(x) d^3x. \tag{45}$$

This volume integral is dependent on the free current density and free charge density. In the following section, we focus on the term of boundary integral since $P(s)$ can be determined straightforward without any difficulty.

In order to clarity the BIEs in $\mathbb{R}^2$, $\mathbb{C}$ and Cl_3, the comparisons are summarized in Table 1. In addition, special care should be taken that the Clifford product are non-commutative. Therefore, results of left and right multiplications are different. Accordingly, the operators are classified as left or right operators.

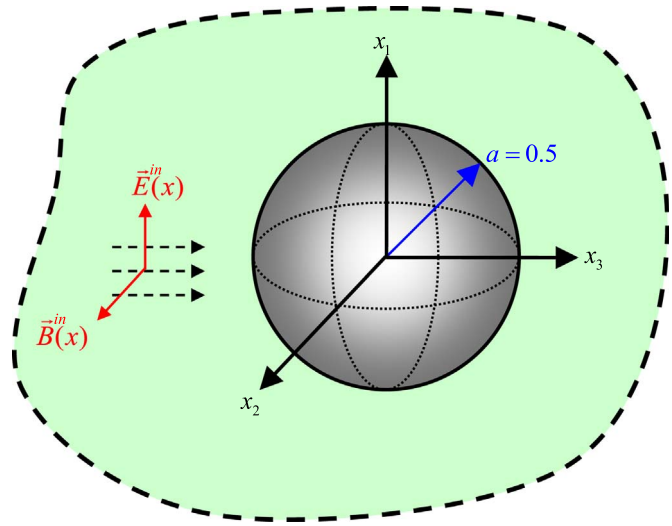


Fig. 1. Sketch of a perfect spherical conductor with a plane electromagnetic wave.

5. Exactly evaluating the Cauchy principal value of Clifford algebra valued boundary integral equation

The Clifford algebra valued BIE for the domain point is

$$f(s) = -\int_{\partial\Omega} C_k(x,s) n(x) f(x) dS(x), \; s \in \Omega. \tag{46}$$

We may encounter the problem of calculating the Cauchy principal value when the source point s is on the boundary. In order to exactly evaluate the Cauchy principal value free of calculating the solid angle, we introduce a semi-analytical technique to deal with the Cauchy principal value.

For this propose, the Clifford algebra valued BIE in Eq. (46) can be divided into two parts. One is the singular part and the other one is the non singular part. Hence, we introduce a simple Clifford algebra valued k-monogenic function $w(x)$ which satisfies

$$\begin{cases} D_k w(x) = 0, \; x \in \Omega, \\ w(s) = f(s). \end{cases} \tag{47}$$

In addition, $w(x)$ also satisfies the condition at infinite

$$\lim_{|x|\to\infty} |w(x)| \le \frac{\beta}{|x|}, \tag{48}$$

since the electromagnetic scattering problem with a perfect conductor is an exterior problem. The value of β is a finite real-valued constant. Here, the simple Clifford algebra valued k-monogenic

Table 1
Comparisons of BIEs among $\mathbb{R}^2$, $\mathbb{C}$ and Cl_3

	BIE for a domain point	Cauchy-type BIE	Weakly-singular kernel	Singular kernel (Cauchy-type kernel)
$\mathbb{R}^2 \to \mathbb{R}$ (Laplace eq.)	$u(x) = -\int_{\partial\Omega} \frac{\partial U(s,x)}{\partial n(s)} u(s) dS(s) + \int_{\partial\Omega} U(s,x) \frac{\partial u(s)}{\partial n(s)} dS(s)$	N.A.	$U(x,s) = -\frac{1}{2\pi} \ln\lvert r\rvert$	$\frac{\partial U(s,x)}{\partial n(s)} = \frac{1}{2\pi} \frac{r \cdot n(s)}{\lvert r\rvert^2}$.
$\mathbb{C} \to \mathbb{C}$(Cauchy-Riemann eq.) (Laplace eq.)	$f(z) = \int_{\partial\Omega} \frac{1}{2\pi i} \frac{1}{s-z} f(s)\, ds + \int_{\partial\Omega} \frac{1}{2\pi i} \ln\lvert s-z\rvert^2 \frac{\partial f(s)}{\partial \bar{s}}\, d\bar{s}$	$f(z) = \int_{\partial\Omega} \frac{1}{2\pi i} \frac{1}{s-z} f(s)\, ds$	$\frac{1}{2\pi i} \ln\lvert s-z\rvert^2$	$\frac{1}{2\pi i} \frac{1}{s-z}$
$\mathbb{R}^3 \to Cl_3$ (Dirac eq.) (Laplace eq.)	$f(s) = -\int_{\partial\Omega} C(x,s) n(x) f(x) dS(x) + \int_{\partial\Omega} U(x,s) n(x) \left(D_x f(x)\right) dS(x)$	$f(s) = -\int_{\partial\Omega} C(x,s) n(x) f(x) dS(x)$	$U(x,s) = \frac{1}{4\pi} \frac{1}{\lvert x-s\rvert}$	$C(x,s) = -\frac{1}{4\pi} \frac{(x-s)}{\lvert x-s\rvert^3}$,
$\mathbb{R}^3 \to Cl_3$ (k-Dirac eq.) (Helmholtz eq.)	$f(s) = -\int_{\partial\Omega} C_k(x,s) n(x) f(x) dS(x) + \int_{\partial\Omega} U(x,s) n(x) \left(D_k f(x)\right) dS(x)$	$f(s) = -\int_{\partial\Omega} C_k(x,s) n(x) f(x) dS(x)$	$U(x,s) = \frac{1}{4\pi} \frac{e^{-ik\lvert x-s\rvert}}{\lvert x-s\rvert}$	$C_k(x,s) = \frac{1}{4\pi}\left[\left(\frac{-ik}{\lvert x-s\rvert} - \frac{1}{\lvert x-s\rvert^2}\right) \frac{e^{-ik\lvert x-s\rvert}}{\lvert x-s\rvert}(x-s) + ik \frac{e^{-ik\lvert x-s\rvert}}{\lvert x-s\rvert}\right]$

function $w(x)$ is chosen as

$$w(x) = \bar{C}_k(x, x_w)\left(\bar{C}_k(s, x_w)\right)^{-1} f(s),\ x \in \Omega \& x_w \in \mathbb{R}^3 \backslash \bar{\Omega}, \tag{49}$$

where

$$\bar{C}_k(x, x_w) = \bar{D}_k U(x, x_w) = \frac{1}{4\pi}\left[\left(\frac{ik}{|x - x_w|} + \frac{1}{|x - x_w|^2}\right)\frac{e^{-ik|x-x_w|}}{|x - x_w|}(x - x_w) + ik\frac{e^{-ik|x-x_w|}}{|x - x_w|}\right]. \tag{50}$$

The Clifford algebra valued BIE for $w(x)$ is

$$w(s) = -\int_{\partial\Omega} C_k(x, s)n(x)w(x)dS(x),\ s \in \Omega. \tag{51}$$

Subtracting Eq. (51) from Eq. (46) and collocating the source point s on the boundary, we have

$$0 = -\int_{\partial\Omega} C_k(x, s)n(x)\left[f(x) - w(x)\right]dS(x),\ s \in \partial\Omega. \tag{52}$$

In this way, the Cauchy principal value can be exactly determined due to the zero boundary density function for $s = x$. In other words, it is no jump term when the collocation point s crosses the boundary. In addition, the flexibility of numerical method can also be preserved. Therefore, this idea is independent of the geometry.

In fact, we chose $x_w = 0$ for all numerical results in this paper. To deal with the unit outward normal vector and the ordinary surface element simultaneously in the numerical implementation, an oriented surface element is introduced into Eq. (52)

$$0 = -\int_{\partial\Omega} C_k(x, s)d\sigma(x)\left[f(x) - w(x)\right],\ s \in \partial\Omega, \tag{53}$$

where $d\sigma(x)$ is the oriented surface element and is defined as

$$n(x)dS(x) = d\sigma(x) = dx_2dx_3e_1 - dx_1dx_3e_2 + dx_1dx_2e_3. \tag{54}$$

In this way, the treatment of normal vectors can be avoided.

6. Matching boundary conditions after discretization

The Dirac matrices [17,19] or so-called the Gamma matrices can produce matrix representation of Clifford algebra $(Cl_{1,3}(\mathbb{C}))$ since they have the relation of specific anticommutation. Therefore, the Dirac matrices are isomorphic to the bases of Clifford algebra $(Cl_{1,3}(\mathbb{C}))$ of grade one. Four Dirac matrices are

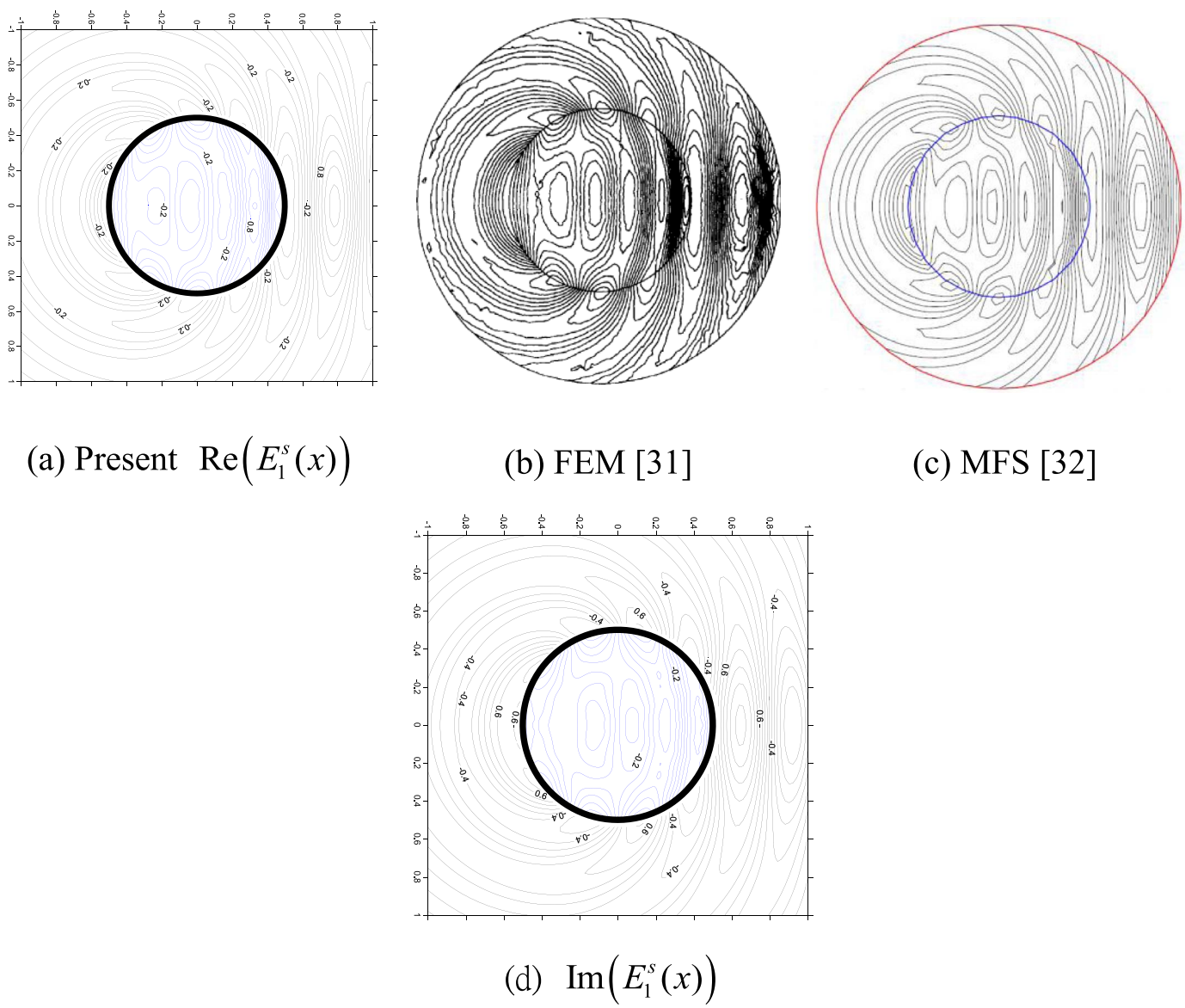
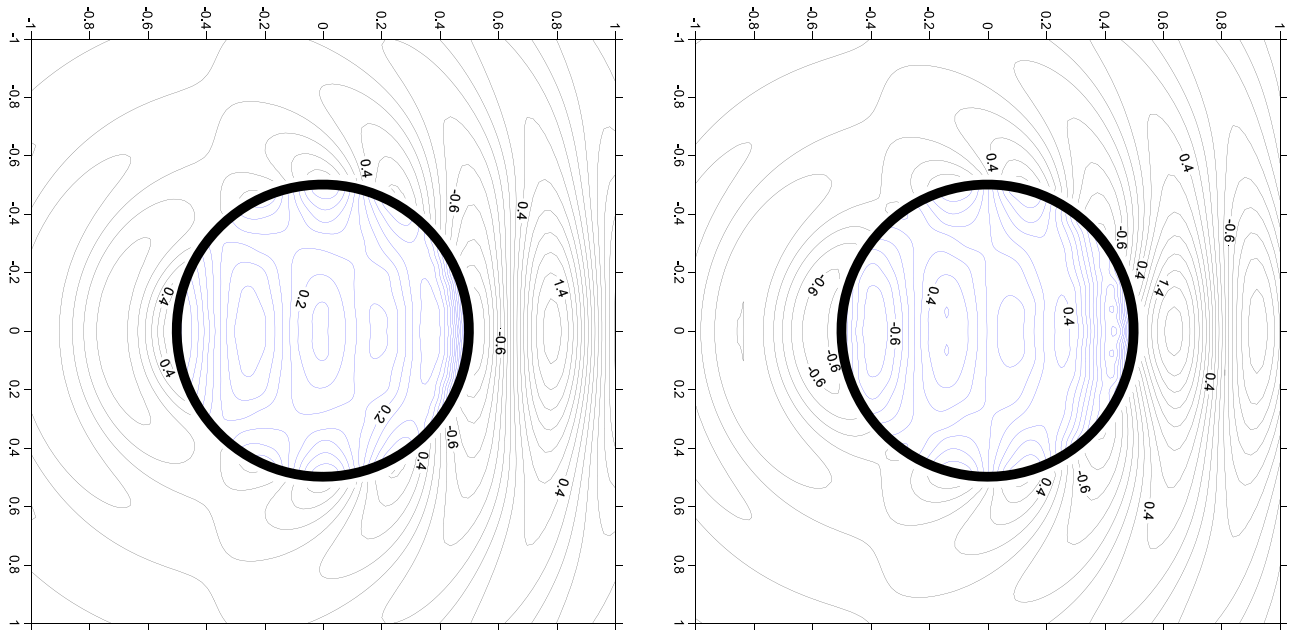
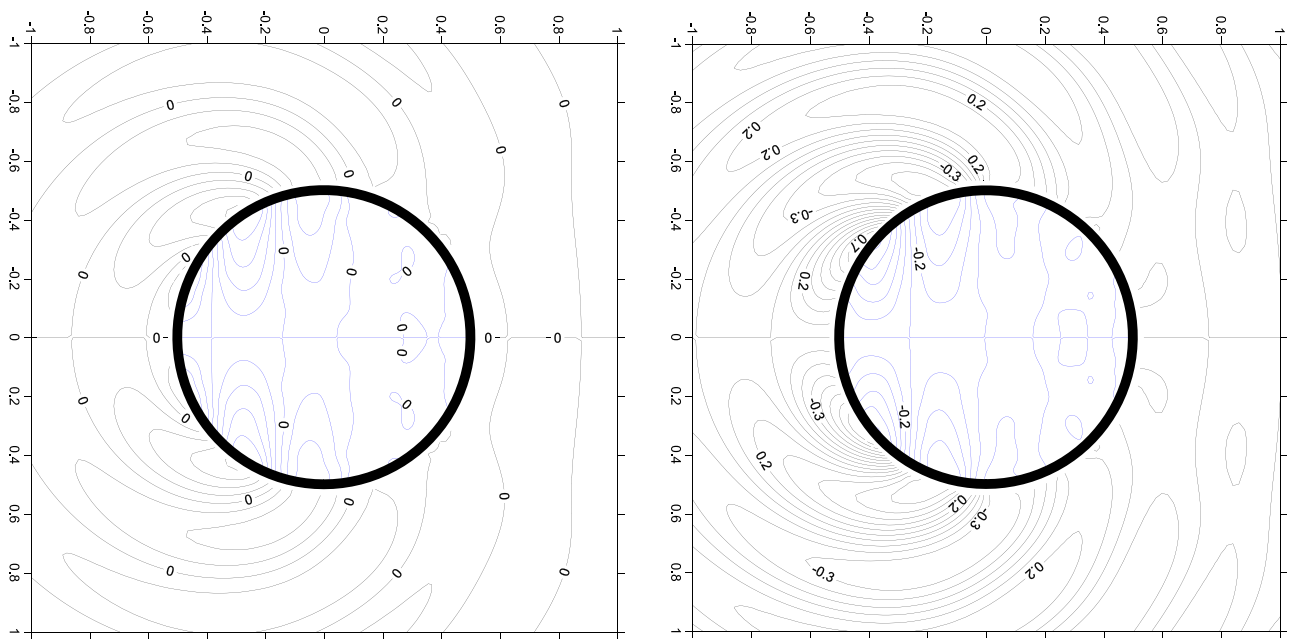


Fig. 2. The contour plots of $E_1^s(x)$ on the $x_1 = 0$ plane and surface of the spherical conductor.



(a) Real-part (b) Imaginary-part

Fig. 3. The contour plots of $B_2^s(x)$ on the $x_1 = 0$ plane and surface of the spherical conductor.



(a) Real-part (b) Imaginary-part

Fig. 4. The contour plots of $B_3^s(x)$ on the $x_1 = 0$ plane and surface of the spherical conductor.

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{55}$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{56}$$

In the basis of the property of the Dirac matrices and the Clifford product rule of $Cl_3(\mathbb{C})$, we can construct eight 4 by 4 complex valued matrices as

$$1 \approx \mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, e_{123} \approx \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \tag{57}$$

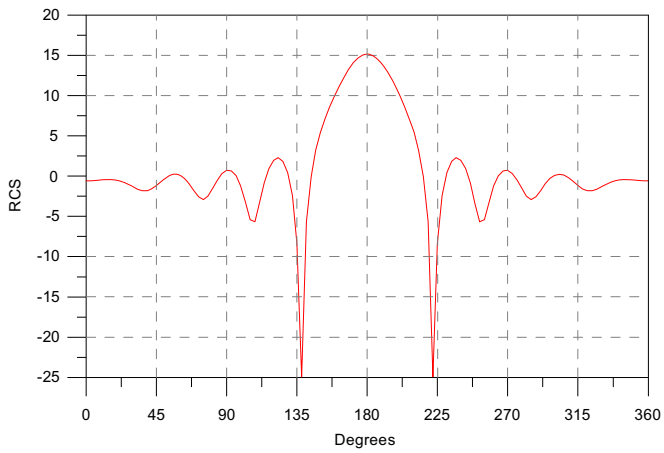


Fig. 5. The RCS of the electric field on the $x_2 = 0$ plane for the spherical conductor.

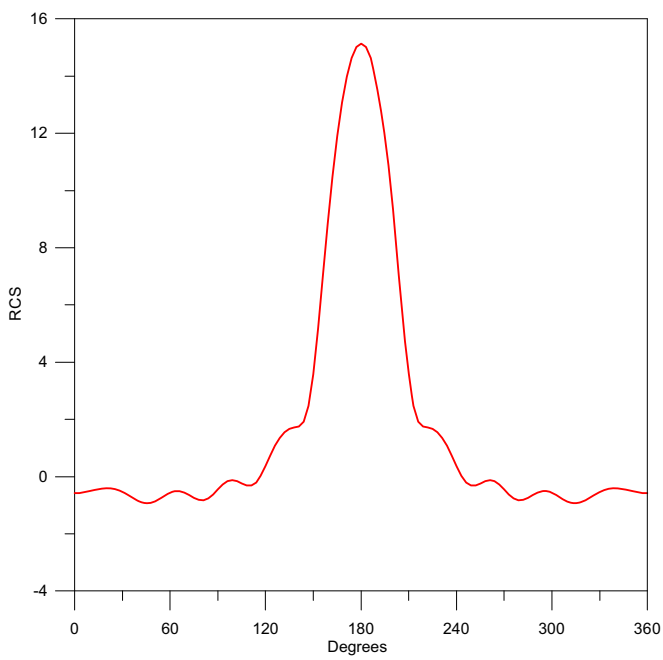


Fig. 6. The RCS of the electric field on the $x_1 = 0$ plane for the spherical conductor.

$$e_1 \approx \gamma^0\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, e_{23} \approx \gamma^2\gamma^3 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix}, \tag{58}$$

$$e_2 \approx \gamma^0\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, e_{31} \approx \gamma^3\gamma^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{59}$$

$$e_3 \approx \gamma^0\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, e_{12} \approx \gamma^1\gamma^2 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}. \tag{60}$$

These eight 4 by 4 complex valued matrices are isomorphic to the bases of the Clifford algebra ($Cl_3(\mathbb{C})$). In the numerical implementation, we adopt eight matrices in Eqs. (57)–(60) to represent the bases of Clifford algebra ($Cl_3(\mathbb{C})$). In this way, the addition and the multiplication of Clifford algebra ($Cl_3(\mathbb{C})$) in the real implementation is nothing but algebra operations of matrix.

Using the constant element scheme to discretize Eq. (53), we have

$$0 = -\sum_{j=1}^{M} \int_{\partial\Omega_j} C_k(x, s_i) d\sigma(x) \left[f_j - w_i(x) \right], s \in \partial\Omega, \tag{61}$$

where the subscript "i" denotes the ith information of collocation point, s_i, M is the number of boundary elements and

$$w_i(x) = \bar{C}_k(x, 0)\left(\bar{C}_k(s_i, 0)\right)^{-1} f_i. \tag{62}$$

In this paper, we employed the triangle Gaussian quadrature to calculate Eq. (61). Then, we have

$$0 = -\sum_{j=1}^{M} \int_{\partial\Omega_j} C_k(x(\xi_1, \xi_2), s_i) d\sigma(x(\xi_1, \xi_2)) \left[f_j - w_i(x(\xi_1, \xi_2)) \right], s \in \partial\Omega, \tag{63}$$

where

$$d\sigma(x(\xi_1, \xi_2)) = -\frac{1}{2} e_{123} \left(\frac{\partial x}{\partial \xi_1} \frac{\partial x}{\partial \xi_2} - \frac{\partial x}{\partial \xi_2} \frac{\partial x}{\partial \xi_1} \right) d\xi_1 d\xi_2. \tag{64}$$

Every point on the flat triangle element can be represented as

$$x(\xi_1, \xi_2) = (1 - \xi_1 - \xi_2) x_a + \xi_1 x_b + \xi_2 x_c, \; 0 \le \xi_1 \le 1, \; 0 \le \xi_2 \le 1, \; \xi_1 + \xi_2 \le 1, \tag{65}$$

in which the vectors x_a, x_b and x_c are the positions of the vertices of an element in terms of Clifford algebra number. It is noted that Eq. (65) is only suitable for the flat element. Therefore, the term, $-\frac{1}{2} e_{123} \left(\frac{\partial x}{\partial \xi_1} \frac{\partial x}{\partial \xi_2} - \frac{\partial x}{\partial \xi_2} \frac{\partial x}{\partial \xi_1} \right)$, in Eq. (64) is a Clifford algebra valued constant. In addition, it is possible to extend the flat element to the curved element. For the curved element, Eq. (65) becomes nonlinear in parametric form and the term in Eq. (64) is also not a Clifford algebra valued constant.

Then, collocating the point on the centroid of each element, we have

$$\left[C_{k(ij)} \right]_{4M \times 4M} \left\{ f_j \right\}_{4M \times 4} = \left\{ 0 \right\}_{4M \times 4}. \tag{66}$$

To fully satisfy the boundary conditions, we change Eq. (66) into the component form as follows

$$\left[C^{comp}_{k(ij)} \right]_{8M \times 8M} \left\{ f^{comp}_{(j)} \right\}_{8M \times 1} = \left\{ 0 \right\}_{8M \times 1}, \tag{67}$$

where $f^{comp}_{(j)}$ is

$$f^{comp}_{(j)} = \left(\left\langle E^{(j)}_0 \;\; E^{(j)}_1 \;\; E^{(j)}_2 \;\; E^{(j)}_3 \;\; B^{(j)}_1 \;\; B^{(j)}_2 \;\; B^{(j)}_3 \;\; B^{(j)}_0 \right\rangle_{8 \times 1} \right)^T. \tag{68}$$

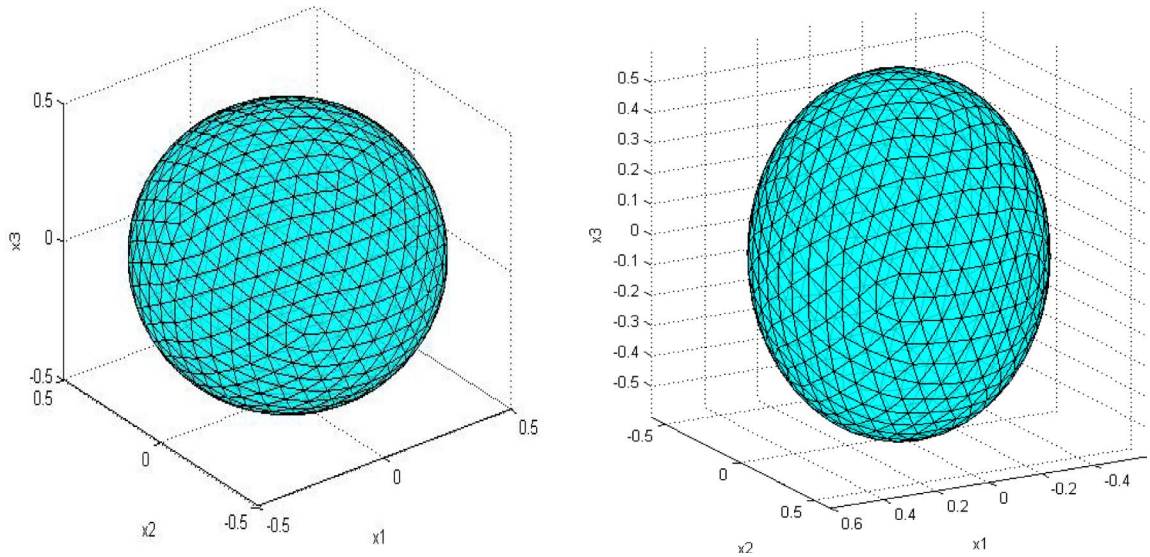
and is the jth boundary density on the jth element. Then, we transform $f^{comp}_{(j)}$ into the local system

$$f^{comp}_{(j)} = \left[\mathrm{T} \right]_{8 \times 8} f^{comp}_{L(j)}, \tag{69}$$

where $f^{comp}_{L(j)}$ is

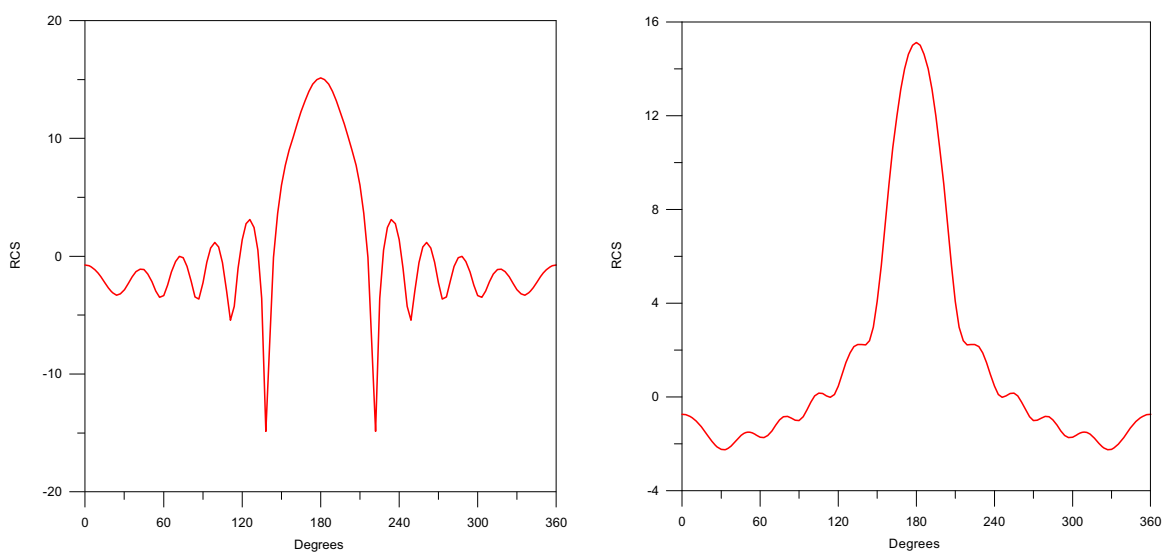
$$f^{comp}_{L(j)} = \left(\left\langle E^{(j)}_0 \;\; E^{(j)}_{n1} \;\; E^{(j)}_{t2} \;\; E^{(j)}_{t3} \;\; B^{(j)}_{n1} \;\; B^{(j)}_{t2} \;\; B^{(j)}_{t3} \;\; B^{(j)}_0 \right\rangle_{8 \times 1} \right)^T, \tag{70}$$

in which $E^{(j)}_{n1}$ and $B^{(j)}_{n1}$ are normal components while $E^{(j)}_{t2}$, $E^{(j)}_{t3}$, $B^{(j)}_{t2}$ and $B^{(j)}_{t3}$ are tangential components. The matrix $\left[\mathrm{T} \right]_{8 \times 8}$ is



(a) Spherical conductor (b) Prolate spheroidal conductor

Fig. 7. The mesh distribution of triangular element for the conductor.



(a) RCS on the $x_2 = 0$ plane (b) RCS on the $x_1 = 0$ plane

Fig. 8. The RCS of the electric field for the prolate spheroidal conductor.

$$[\mathrm{T}]_{8\times 8} = \begin{bmatrix} 1 & & & \\ & R^T & & \\ & & R^T & \\ & & & 1 \end{bmatrix}, \tag{71}$$

where R is a rotation matrix

$$R = \begin{bmatrix} n_1 & n_2 & n_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}, \tag{72}$$

in which two vectors $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ are obtained by

$$(\beta_1, \beta_2, \beta_3) = -\frac{(n_1, n_2, n_3) \times \vec{E}^{in}(x)}{\left|(n_1, n_2, n_3) \times \vec{E}^{in}(x)\right|}, \tag{73}$$

$$(\gamma_1, \gamma_2, \gamma_3) = \frac{(n_1, n_2, n_3) \times (\beta_1, \beta_2, \beta_3)}{\left|(n_1, n_2, n_3) \times (\beta_1, \beta_2, \beta_3)\right|}. \tag{74}$$

In general, $E_0^{(j)}$ and $B_0^{(j)}$ are zero since the function $f(x)$ in Eq. (33) only has six components. For the case of electromagnetic scattering problem with a perfect conductor, $E_{t2}^{(j)}$, $E_{t3}^{(j)}$ and $B_{n1}^{(j)}$ are specified while $E_{n1}^{(j)}$, $B_{t2}^{(j)}$ and $B_{t3}^{(j)}$ are unknown. After satisfying the boundary conditions in Eqs. (15)–(16) and arranging Eq. (67), the unknowns can be obtained.

7. Illustrative examples

7.1. Electromagnetic scattering with a perfect spherical conductor

In order to demonstrate the validity of the present approach, an electromagnetic scattering problem with a perfect spherical

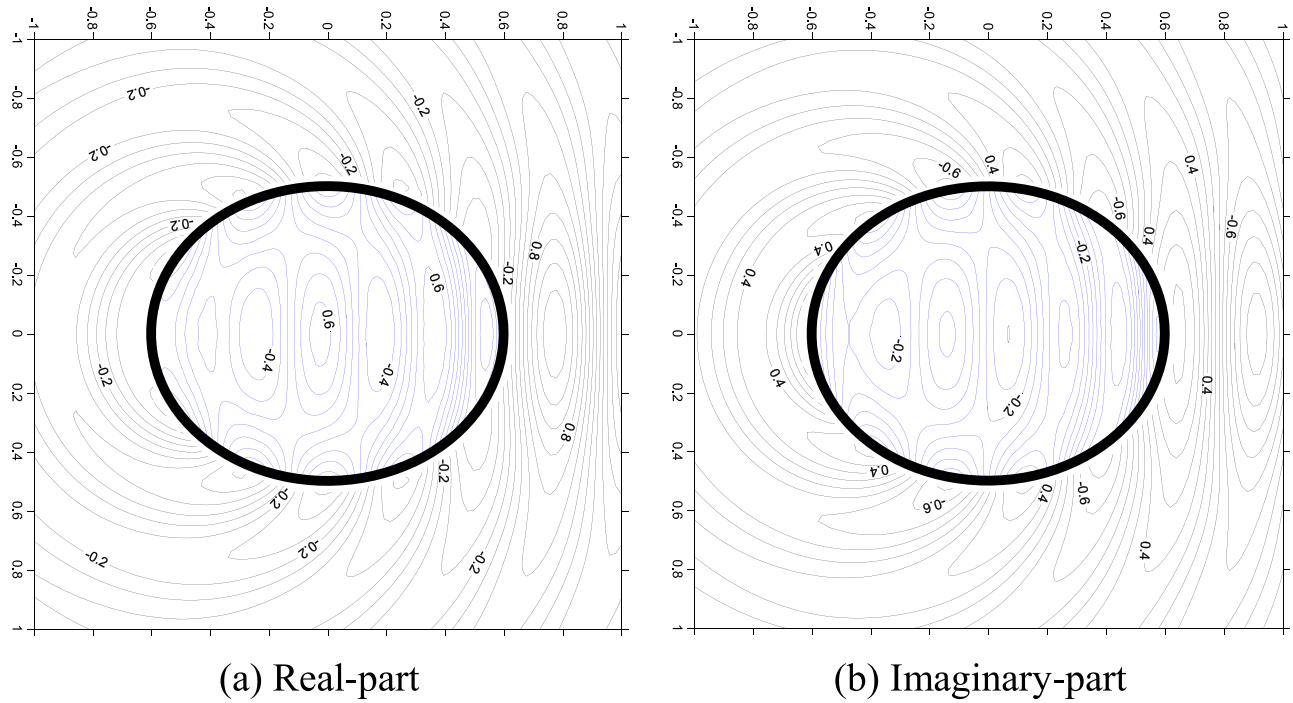


Fig. 9. The contour plots of $E_1^s(x)$ on the $x_1 = 0$ plane and surface of the prolate spheroidal conductor.

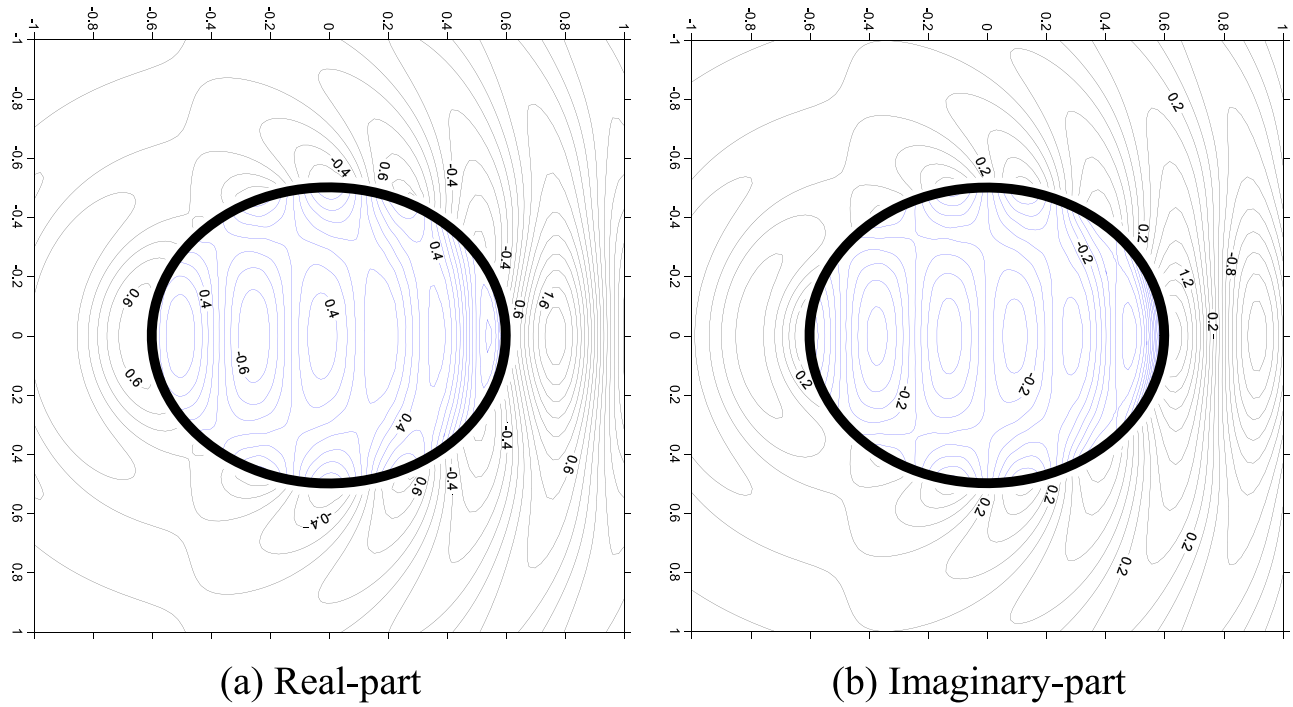


Fig. 10. The contour plots of $B_2^s(x)$ on the $x_1 = 0$ plane and surface of the prolate spheroidal conductor.

conductor is considered as shown in Fig. 1. Here, we consider no free current density and free charge density in this example. The radius of the conductor is $a = 0.5$ m. The values of wave number $k = 4\pi$ m^{-1}, permittivity, $\varepsilon = 1$ A^2·s^4·kg^{-1}·m^{-3}, and permeability, $\mu = 1$ A·s^{-2}·kg·m, are considered. The incident wave is a uniform plane electromagnetic wave

$$\vec{E}^{in}(x) = (e^{-ikx_3}, 0, 0), \tag{75}$$

$$\vec{B}^{in}(x) = (0, e^{-ikx_3}, 0). \tag{76}$$

The incident wave propagates in the positive x_3 direction and normally incident upon the perfectly conductor. Therefore, the incident wave in terms of the language of Clifford algebra in $Cl_3(\mathbb{C})$ is

$$f_{in}(x) = e^{-ikx_3}e_1 + ce^{-ikx_3}\, e_{31},\ x \in \Omega, \tag{77}$$

where c is 1 m·s^{-1} in this case. The contour plots of the component

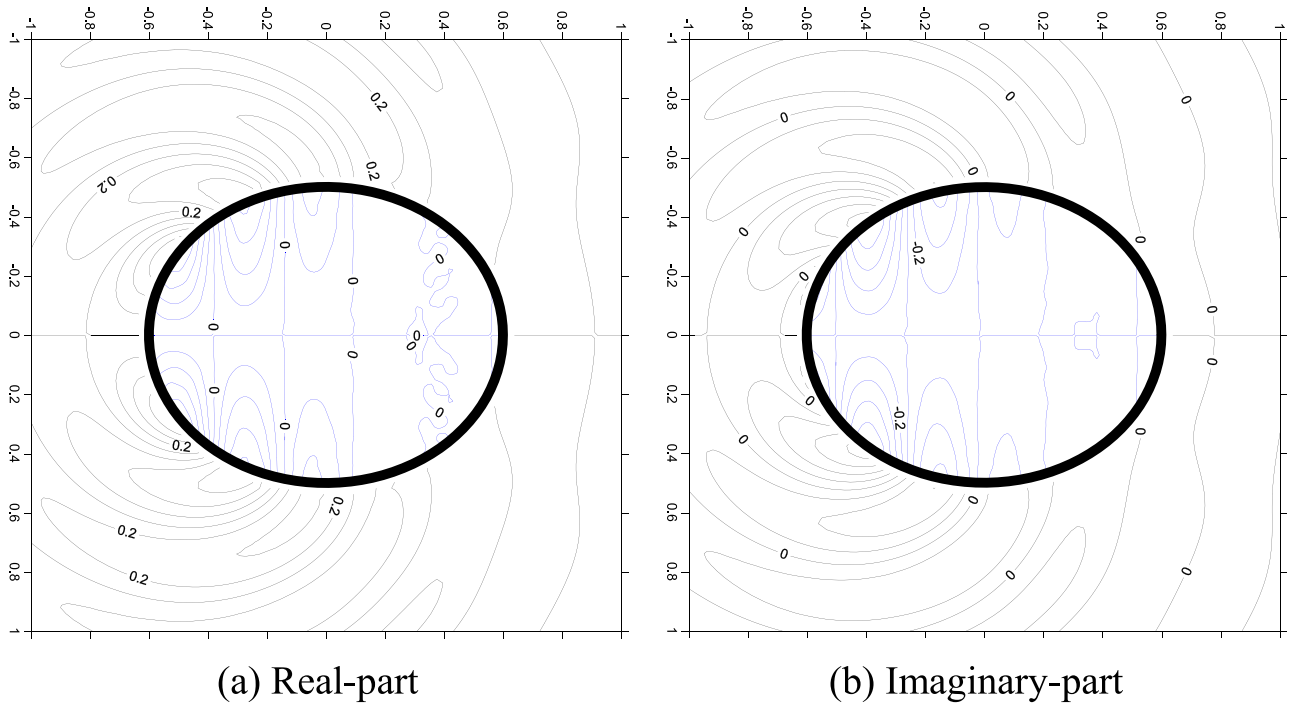


Fig. 11. The contour plots of $B_3^s(x)$ on the $x_1 = 0$ plane and surface of the prolate spheroidal conductor.

in the x_1 direction of scattered electric field $E_1^s(x)$ on the $x_1 = 0$ plane and surface of the spherical conductor are shown in Fig. 2. It is noted that the contour value inside the circle is the corresponding scattered electric field on the surface of spherical conductor for $x_1 \geq 0$. After comparing with those of results obtained from FEM [34] and MFS [35], the present results are acceptable. In addition, the contour plots of $B_2^s(x)$ and $B_3^s(x)$ on the $x_1 = 0$ plane and surface of the spherical conductor are displayed in Figs. 3 and 4, respectively. Since we use the language of Clifford algebra to deal with Maxwell's equation, the electric and the magnetic flux density fields can be obtained simultaneously without any differential and integral operations. For the far-field response, the present results of the radar cross sections (RCS) on the $x_2 = 0$ and the $x_1 = 0$ planes are shown in Figs. 5 and 6, respectively. The definition of the RCS in the three dimensional case is

$$\left\{ \begin{matrix} \sigma_E \\ \sigma_B \end{matrix} \right. = \begin{cases} \lim\limits_{|x|\to\infty} \left[4\pi |x|^2 \dfrac{\left|\vec{E}^s(x)\right|^2}{\left|\vec{E}^{in}(x)\right|^2} \right], \\ \lim\limits_{|x|\to\infty} \left[4\pi |x|^2 \dfrac{\left|\vec{B}^s(x)\right|^2}{\left|\vec{B}^{in}(x)\right|^2} \right]. \end{cases} \tag{78}$$

The real computational quantity of RCS is

$$\mathrm{RCS} = 10 \log_{10}\left(\sigma_{\mathrm{EorB}}\right) \tag{79}$$

In the numerical implementation, we chose $|x|$ to be equal to 10. Totally, 642 nodal points with 1280 elements were used to obtain all numerical results. The mesh of real distribution of triangular element is shown in Fig. 7(a). It is found that the results of the present approach, FEM and MFS match well. In addition, the RCS of scattered electric and magnetic flux density fields are the same, owing to the relationship of

$$\left|\vec{E}(x)\right|^2 = c^2 \left|\vec{B}(x)\right|^2. \tag{80}$$

Therefore, only the RCS of scattered electric field is provided in this example. For the results of MFS, they used 1344 nodal points. In the FEM formulation, 99,991 nodal points, 589,505 elements and 706,999 edges were employed [34]. This is reason why the BEM is efficient for dealing with exterior problems.

7.2. Electromagnetic scattering with a perfect prolate spheroidal conductor

A prolate spheroidal conductor is also considered in this example. The prolate spheroidal is described by

$$\left(\frac{x_1}{0.5}\right)^2 + \left(\frac{x_2}{0.5}\right)^2 + \left(\frac{x_3}{0.6}\right)^2 = 1, \tag{81}$$

and the figure of mesh distribution is shown in Fig. 7(b). The values of corresponding parameters and the incident wave are the same with the case of spherical conductor. Fig. 8 shows the RCS on the $x_2 = 0$ and the $x_1 = 0$ plane and contours of the scattered electric field $E_1^s(x)$ on the $x_1 = 0$ plane and surface of the prolate spheroidal conductor are shown in Fig. 9. It is also noted that the contour value inside the ellipse is the corresponding scattered electric field on the surface of prolate spheroidal conductor for $x_1 \geq 0$. Good agreement is made after comparing above results with those of MFS [35]. Furthermore, we also show the contour plots of scattered magnetic flux density field $\vec{B}_2^s(x)$ and $\vec{B}_3^s(x)$ on the $x_1 = 0$ plane and surface of the prolate spheroidal conductor in Figs. 10 and 11, respectively.

8. Conclusions

Through the use of Clifford analysis in $Cl_3(\mathbb{C})$, the time harmonic Maxwell's equations are found nothing but the nonhomogeneous k-Dirac equation. It is remarkable that both are of the first order. The nonhomogeneous k-Dirac equation is more elegant and

simpler than Maxwell's equations not only for the formulation but also for the numerical implementation. Therefore, the electromagnetic scattering problem can be straightforward solved by using the Clifford algebra valued BIE with the Cauchy type kernels. In this way, the unknown function is a Clifford algebra valued function consisted of electric field and the magnetic flux density. Hence, they can be solved simultaneously.

To exactly evaluate the Cauchy principal value, a simple Clifford algebra valued k-monogenic function is considered. Therefore, calculating the solid angle is free. Since this way is suitable for any geometry shape, it can preserve the flexibility of numerical method. For the term of remaining boundary integral, it can be calculated by using the triangle Gaussian quadrature without any difficulty. The unit outward normal vector can be combined with the ordinary surface element by introducing an oriented surface element.

In addition, all mathematical operations of Clifford algebra in the numerical implementation can be achieved by introducing the Dirac matrices. In order to match the boundary conditions, we discretized the Clifford algebra valued BIE into a Clifford algebra valued BEM and transformed the global boundary densities into the local boundary densities. Finally, to demonstrate the validity of Clifford algebra valued BEM, two electromagnetic scattering problems with the perfect spherical conductor and the prolate spheroidal conductor are both considered. Results show the accuracy of the Clifford algebra valued BEM after comparing with those of FEM and MFS.

Acknowledgement

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