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Cylindrical and spherical inflation in compressible finite elasticity

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Murphy (1992) examined cylindrical and spherical inflation of compressible perfectly elastic materials having three special forms of the strain-energy function. In this paper a general procedure for handling such problems for any strain-energy function is proposed. This procedure is used to confirm some of the results by Murphy as well as to deduce new solutions. One solution obtained by that author for cylindrical inflation is found to be incorrect.

1. Introduction

In a recent article appearing in this journal, Murphy (1992) examined cylindrical and spherical inflation in compressible finite elasticity for three special strain-energy functions, which are termed materials of types IV, V, and VI. Using a substitution first exploited by Chung *et al.* (1986) a number of closed-form solutions are deduced for spherical inflation and eversion while the basic ordinary differential equations are presented, but not integrated, for cylindrical inflation. The purpose of this paper is to show that a more elegant procedure exists for handling such ordinary differential equations which can be used to confirm the results obtained by Murphy (1992), as well as to deduce new integrals of equations formulated by that author. One solution obtained by Murphy for cylindrical inflation is found to be incorrect (see equation (4.19) below). Moreover, this procedure can, at least in principle, be utilized for cylindrical and spherical inflation problems for any strain-energy function.

The compressible finite elastic materials studied by Murphy (1992) have strain-energy functions given by

$$\left.\begin{array}{ll} \text{(IV)} & W = c_1 i_1 i_2 + c_2 i_1 + c_3 i_2 + c_4 i_3 + c_5, \\ \text{(V)} & W = c_1 i_2 i_3 + c_2 i_1 + c_3 i_2 + c_4 i_3 + c_5, \\ \text{(VI)} & W = c_1 i_1 i_3 + c_2 i_1 + c_3 i_2 + c_4 i_3 + c_5, \end{array}\right\} \tag{1.1}$$

where $c_1, \dots, c_5$ denote material constants which are required to satisfy a number of restrictions which are detailed in Murphy (1992). Further, in equation (1.1), i_1, i_2, and i_3 denote the principal invariants of the right or left stretch tensors $\boldsymbol{U}$ or $\boldsymbol{V}$ in the usual polar decomposition of the deformation gradient tensor $\boldsymbol{F}$, that is

$$\boldsymbol{F} = \boldsymbol{RU} = \boldsymbol{VR}, \tag{1.2}$$

where $\boldsymbol{R}$ is a proper orthogonal tensor representing the rotation.

In terms of the usual cylindrical and spherical polar coordinates (R, Θ, Z) and (R, Θ, Φ) respectively, the axially symmetric cylindrical and spherical deformations are given respectively by

$$r = r(R), \qquad \theta = \Theta, \qquad z = \gamma Z, \tag{1.3}$$

$$r = r(R), \qquad \theta = \Theta, \qquad \phi = \Phi, \tag{1.4}$$

with the usual convention for material and spatial coordinates. Here γ is a constant and in each case $r(R)$ is a function of R only. For a proper theory, which must be invariant under changes of the length scale, the second-order ordinary differential equation for $r(R)$ is formally invariant under the stretching one-parameter group of transformations

$$r_1 = \lambda r, \qquad R_1 = \lambda R, \tag{1.5}$$

for arbitrary constants λ. This means that the second-order ordinary differential equation for $r(R)$, no matter how nonlinear, can always be reduced to an ordinary differential equation of first order. Specifically the sequence of transformations (and starting with the usual Euler transformation)

$$R = \mathrm{e}^t, \qquad u = \frac{r}{R}, \qquad p = \frac{\mathrm{d}u}{\mathrm{d}t} \tag{1.6}$$

will always generate a first-order differential equation for p as a function of u. This is in contrast to the device used by Murphy (1992), and first exploited by Chung *et al.* (1986), which does not always produce a first-order ordinary differential equation.

In the following section we briefly illustrate (1.6) for spherical inflation of materials of types IV and V which gives rise to the expression obtained by Murphy (1992) for the materials of type IV and V. In the subsequent section we apply the procedure to the equations given by Murphy for spherical eversion of materials of types IV, V, and VI and deduce some new special solutions. In the final section of the paper we examine the problem of cylindrical inflation and fully integrate Murphy's equations.

2. Illustration of method for spherical inflation

In this section we briefly show that the procedure proposed here generates the solutions presented in Murphy (1992) for spherical inflation for materials of types IV and V. For spherical inflation (1.4) of a material of type IV the basic governing differential equation arising from the equilibrium equations becomes

$$2r\left(R^2\frac{\mathrm{d}^2r}{\mathrm{d}R^2} + R\frac{\mathrm{d}r}{\mathrm{d}R}\right) + R^2\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2 - 3r^2 = 0, \tag{2.1}$$

and, on making the Euler transformation $R = \mathrm{e}^t$, we have

$$2r\frac{\mathrm{d}^2r}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - 3r^2 = 0. \tag{2.2}$$

Further, on making the transformation $r = e^t u$, we obtain

$$2u\frac{d^2u}{dt^2} + 6u\frac{du}{dt} + \left(\frac{du}{dt}\right)^2 = 0, \tag{2.3}$$

for which either we can use

$$p = \frac{du}{dt}, \qquad \frac{d^2u}{dt^2} = p\frac{dp}{du}, \tag{2.4}$$

and integrate the resulting first-order ordinary differential equation or we can integrate (2.3) directly by simply rewriting the equation as

$$\frac{u_{tt}}{u_t} + 3 = -\frac{u_t}{2u}, \tag{2.5}$$

where we are using the suffix notation for derivatives. A straightforward integration of (2.5) yields

$$u^{1/2}\frac{du}{dt}e^{3t} = -2A, \tag{2.6}$$

where A is a constant and a further integration gives

$$u^{3/2} = Ae^{-3t} + B, \tag{2.7}$$

where B is the second integration constant. From (2.7) we may deduce

$$r^3 = (A + BR^3)^2/R^3, \tag{2.8}$$

which agrees with the result obtained by Murphy (1992).

Similarly, for spherical inflation of a material of type V, the basic governing differential equation is

$$2r\left(R^2\frac{d^2r}{dR^2} - R\frac{dr}{dR}\right) + 3R^2\left(\frac{dr}{dR}\right)^2 - r^2 = 0, \tag{2.9}$$

and an application of the above procedure gives

$$2u\frac{d^2u}{dt^2} + 6u\frac{du}{dt} + 3\left(\frac{du}{dt}\right)^2 = 0. \tag{2.10}$$

Integration of this equation gives

$$r^5 = (A + BR^3)^2/R, \tag{2.11}$$

which agrees with Murphy's solution.

We note that an important feature of equations (2.1) and (2.9) is that they both admit the homogeneous deformation $r = AR$, which is the essential reason why the r^2 term disappears from the subsequent equations (2.3) and (2.10) respectively. The equations of the following section do not enjoy this particular property, but can still be reduced to an ordinary differential equation of first order.

3. Spherical eversion

Following Murphy (1992), we consider spherical eversion for materials of types IV, V, and VI. Thus we consider the deformation

$$r = r(R), \qquad \theta = \pi - \Theta, \qquad \phi = \Phi, \tag{3.1}$$

where, for materials of type IV, the function $r(R)$ satisfies the ordinary differential equation

$$2r\left(R^2 \frac{d^2 r}{dR^2} + R\frac{dr}{dR}\right) + R^2\left(\frac{dr}{dR}\right)^2 - 3r^2 = \alpha R^2 + \beta r R, \tag{3.2}$$

where α and β denote new material constants defined by

$$\alpha = 2c_2/c_1, \qquad \beta = 2c_3/c_1. \tag{3.3}$$

The procedure described in the previous section using the sequence of transformations (1.6) gives

$$p\left(2u\frac{dp}{du} + p + 6u\right) = \alpha + \beta u. \tag{3.4}$$

For α and β nonzero, this equation is not readily solved except in two special cases. Firstly, if the material constants are such that $\alpha = \frac{1}{4}\beta^2$, that is

$$c_3^2 = 2c_1c_2, \tag{3.5}$$

then (3.4) admits the special solution

$$p = -(\tfrac{1}{2}\beta + 2u), \tag{3.6}$$

and it is a simple matter to show that this special solution corresponds to

$$r = \frac{A}{R} - \tfrac{1}{4}\beta R, \tag{3.7}$$

where A denotes an arbitrary constant. Secondly, if the material constants are such that $\alpha = \frac{1}{36}\beta^2$, that is,

$$c_3^2 = 18c_1c_2, \tag{3.8}$$

then (3.4) admits the special solution $p = \alpha^{1/2}$, so that in this case the corresponding solution of (3.2) becomes

$$r = \alpha^{1/2} R \log(R/R_0), \tag{3.9}$$

where R_0 denotes a constant.

For materials of type V, the basic ordinary differential equation given by Murphy is

$$2r^3\left(R^2 \frac{d^2 r}{dR^2} - R\frac{dr}{dR}\right) + 3r^2R^2\left(\frac{dr}{dR}\right)^2 + r^4 = \alpha R^4 + \beta r R^3, \tag{3.10}$$

and it is important to note that in this case the left-hand side is different from equation (2.9) and accordingly the terms not involving derivatives in the equation for u do not cancel and we obtain

$$2u^3\frac{d^2u}{dt^2}+6u^3\frac{du}{dt}+3u^2\left(\frac{du}{dt}\right)^2+2u^4=\alpha+\beta u. \tag{3.11}$$

This equation can still be reduced to an ordinary differential equation of first order but is not solvable in general for α and β nonzero. For α and β zero, the solution given by Murphy can be deduced from (3.11) by means of the transformation $v=u^{5/2}$.

For materials of type VI, the differential equation for $r(R)$ is

$$r^2R^2\frac{d^2r}{dR^2}+rR^2\left(\frac{dr}{dR}\right)^2+r^3=\alpha R^3+\beta rR^2, \tag{3.12}$$

and the corresponding equation for u is

$$u^2\frac{d^2u}{dt^2}+3u^2\frac{du}{dt}+u\left(\frac{du}{dt}\right)^2+2u^3=\alpha+\beta u, \tag{3.13}$$

which can be fully integrated for α zero as follows. From (3.13), and $v=u^2$ we have

$$\frac{d^2v}{dt^2}+3\frac{dv}{dt}+4v=2\beta, \tag{3.14}$$

which integrates to give

$$v=\tfrac{1}{2}\beta+Ce^{-3t/2}\sin\left(\tfrac{1}{2}\sqrt{7}\,t+\varepsilon\right), \tag{3.15}$$

where C and ε denote arbitrary constants, and from (3.15) we may deduce

$$r^2=\tfrac{1}{2}\beta R^2+CR^{1/2}\sin\left(\tfrac{1}{2}\sqrt{7}\log R+\varepsilon\right), \tag{3.16}$$

and this is marginally more general than the result obtained by Murphy, which is originally due to Armanni (1915).

In the final section of the paper, we are able to fully integrate the equations proposed by Murphy.

4. Cylindrical inflation

For cylindrical inflation given by equation (1.3), the basic equation for a material of type IV is

$$2R\left(R^2\frac{d^2r}{dR^2}+R\frac{dr}{dR}-r\right)+\frac{1}{\gamma}\left[2rR^2\frac{d^2r}{dR^2}+R^2\left(\frac{dr}{dR}\right)^2-r^2\right]=0, \tag{4.1}$$

where we have corrected a term in the corresponding equation given by Murphy. (This is apparent since from Ericksen (1955) all compressible finite elastic materials must admit the homogeneous deformation $r=AR$, which is not the case for Murphy's equation (8.4).) On transforming equation (4.1), we find

$$2\left(\frac{d^2u}{dt^2}+2\frac{du}{dt}\right)+\frac{1}{\gamma}\left[2u\frac{d^2u}{dt^2}+4u\frac{du}{dt}+\left(\frac{du}{dt}\right)^2\right]=0, \tag{4.2}$$

which can be rearranged to give

$$(\gamma+u)\left(\frac{d^2u}{dt^2}+2\frac{du}{dt}\right)=-\tfrac{1}{2}\left(\frac{du}{dt}\right)^2. \tag{4.3}$$

On rewriting this equation as

$$\frac{u_{tt}}{u_t}+2=-\frac{u_t}{2(\gamma+u)}, \tag{4.4}$$

we may deduce the first ingeral

$$\frac{du}{dt}e^{2t}(\gamma+u)^{1/2}=-\tfrac{4}{3}A, \tag{4.5}$$

where A is a constant. A further integration yields

$$(\gamma+u)^{3/2}=(A+BR^2)/R^2, \tag{4.6}$$

so that altogether we have

$$r=R^{1/3}\left(\frac{A}{R}+BR\right)^{2/3}-\gamma R, \tag{4.7}$$

which is the required general solution of (4.1).

For materials of type V, the basic differential equation given by Murphy is

$$\frac{2r}{R}\left[rR^2\frac{d^2r}{dR^2}+R^2\left(\frac{dr}{dR}\right)^2-rR\frac{dr}{dR}\right]+\gamma\left[2rR^2\frac{d^2r}{dR^2}+R^2\left(\frac{dr}{dR}\right)^2-r^2\right]=0, \tag{4.8}$$

which simplifies eventually to give

$$u\frac{d^2u}{dt^2}+2u\frac{du}{dt}+\left(\frac{du}{dt}\right)^2=\frac{\gamma}{2(\gamma+u)}\left(\frac{du}{dt}\right)^2. \tag{4.9}$$

On making the substitution $v=u^2$ we have

$$\frac{v_{tt}}{v_t}+2=\tfrac{1}{4}\gamma\frac{v_t}{v(\gamma+v^{1/2})}, \tag{4.10}$$

which on integrating gives

$$\log v_t+2t=\tfrac{1}{4}\gamma\int\frac{dv}{v(\gamma+v^{1/2})}=\tfrac{1}{2}\gamma\int\frac{du}{u(\gamma+u)}, \tag{4.11}$$

and therefore we have

$$\frac{dv}{dt}e^{2t}\left(\frac{\gamma+u}{u}\right)^{1/2}=-4A. \tag{4.12}$$

A further integration yields

$$\int[u(\gamma+u)]^{1/2}\,du=Ae^{-2t}+B. \tag{4.13}$$

The integral can be evaluated to eventually give

$$\tfrac{1}{2}\{(u+\tfrac{1}{2}\gamma)[u(\gamma+u)]^{1/2}-\tfrac{1}{4}\gamma^2\cosh^{-1}(1+2u/\gamma)\}=\frac{A}{R^2}+B, \tag{4.14}$$

noting also that $u=r/R$.

For materials of type VI, Murphy's equation

$$2rR^2\frac{d^2r}{dR^2}+R^2\left(\frac{dr}{dR}\right)^2-r^2=0, \tag{4.15}$$

transforms to give

$$\frac{u_{tt}}{u_t}+2=-\frac{u_t}{2u}, \tag{4.16}$$

which integrates to yield

$$\frac{du}{dt}e^{2t}u^{1/2}=-\tfrac{4}{3}A, \tag{4.17}$$

so that

$$u^{3/2}=Ae^{-2t}+B. \tag{4.18}$$

In terms of the original variables we have

$$\left(\frac{r}{R}\right)^{3/2}=\frac{A}{R^2}+B, \tag{4.19}$$

which differs considerably from Murphy's result. We observe that for a solid cylinder the constant A is zero for each of the three deformations (4.7), (4.14), and (4.19) and the homogeneous deformation $r=\lambda R$ is recovered in each case, where λ denotes a constant.

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