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Some new closed-form solutions describing spherical inflation in compressible finite elasticity

JEREMIAH G. MURPHY†
Department of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD, UK

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Within the context of finite elasticity, considerable interest has been shown in the problem of inflating spherical shells of special compressible materials. Three new closed-form solutions to the above problem are presented. The qualitative features of the inflation of thin shells of these special materials are then studied. The related problems of spherical eversion and cylindrical inflation are also briefly considered.

0. Introduction

The problem of spherical inflation and compaction of shells of special compressible materials has been studied in a number of recent papers. For the harmonic material introduced by John [1], a closed-form solution to the equations of equilibrium describing spherical inflation and compaction was presented by Abeyaratne & Horgan [2] and independently by Ogden [3]. A similar solution for a special Blatz–Ko material was obtained by Chung *et al.* [4]. In his major contribution, Carroll [5] found solutions describing spherical inflation and compaction for three quite general compressible elastic materials. One of these materials is the harmonic material and another was introduced independently by Haughton [6], who also considered spherical inflation and compaction.

Each of the four solutions mentioned above is controllable in the sense that the form of the deformation field is independent of the specific form of the strain energy function. We will show how the equations of equilibrium themselves motivate six forms of the strain energy function for which controllable deformations describing spherical inflation and compaction exist. Of these six materials, three are those studied by Carroll [5], while a fourth is a generalization of a strain energy function introduced by Armanni [7]. The other two strain energy functions appear to be new. With the aid of a substitution introduced by Chung *et al.* [4], closed-form solutions to the equilibrium equations will be obtained for the three materials introduced here.

In a recent review article, Beatty [8] examined in detail the inflation of thin shells (or balloons) having special forms of the strain energy function introduced by Blatz & Ko [9]. The inflation of thin shells of the materials introduced here will be studied and the qualitative features compared with the features of shells of the special Blatz–Ko material studied by Chung *et al.* [4].

† Present address: Department of Mathematical Physics, University College Cork, Cork, Ireland.

Armanni [7] also considered the problem of spherical eversion. Motivated by this, we will study the eversion of spherical shells of the materials introduced here. Cylindrical inflation and compaction will be also briefly discussed.

1. Preliminaries

The response of an elastic material is described completely by the form of its strain energy function

$$W = \hat{W}(\boldsymbol{F}), \tag{1.1}$$

where $\boldsymbol{F}$ is the deformation gradient tensor satisfying

$$\det \boldsymbol{F} > 0. \tag{1.2}$$

We note that $\boldsymbol{F}$ has the polar decompositions

$$\boldsymbol{F} = \boldsymbol{RU} = \boldsymbol{VR}, \tag{1.3}$$

where the rotation $\boldsymbol{R}$ is a proper orthogonal tensor and the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are positive-definite and symmetric.

Invariance under rigid-body motions leads to

$$W = \tilde{W}(\boldsymbol{U}). \tag{1.4}$$

The assumption of material isotropy further leads to

$$W = \bar{W}(i_1, i_2, i_3), \tag{1.5}$$

where i_1, i_2, and i_3 are the principal invariants of $\boldsymbol{U}$ (and of $\boldsymbol{V}$, since $\boldsymbol{U}$ and $\boldsymbol{V}$ have identical invariants).

The stress response equations

$$\boldsymbol{P} = \frac{\partial W}{\partial \boldsymbol{F}}, \qquad \boldsymbol{T} = i_3^{-1}\boldsymbol{P}\boldsymbol{F}^{\mathrm{T}}, \tag{1.6}$$

where $\boldsymbol{P}$ and $\boldsymbol{T}$ are the Piola and Cauchy stress tensors, then lead to a representation for isotropic materials,

$$\boldsymbol{T} = i_3^{-1}\left(i_2\frac{\partial W}{\partial i_2} + i_3\frac{\partial W}{\partial i_3}\right)\mathbf{1} + i_3^{-1}\frac{\partial W}{\partial i_1}\boldsymbol{V} - \frac{\partial W}{\partial i_2}\boldsymbol{V}^{-1}, \tag{1.7}$$

on application of the Cayley–Hamilton theorem.

Substitution of (1.7) into the equations of equilibrium

$$\operatorname{div} \boldsymbol{T} = \mathbf{0}, \tag{1.8}$$

leads to the form

$$\begin{aligned}
&\frac{\partial W}{\partial i_1}\operatorname{div}(i_3^{-1}\boldsymbol{V}) + \frac{\partial W}{\partial i_2}\operatorname{div}(\operatorname{tr}\boldsymbol{V}^{-1}\mathbf{1} - \boldsymbol{V}^{-1}) + \frac{\partial^2 W}{\partial i_1^2}i_3^{-1}\boldsymbol{V}\nabla i_1 \\
&+ \frac{\partial^2 W}{\partial i_1\,\partial i_2}(\operatorname{tr}\boldsymbol{V}^{-1}\nabla i_1 - \boldsymbol{V}^{-1}\nabla i + i_3^{-1}\boldsymbol{V}\nabla i_2) + \frac{\partial^2 W}{\partial i_1\,\partial i_3}(i_3^{-1}\boldsymbol{V}\nabla i_3 + \nabla i_1) \\
&+ \frac{\partial^2 W}{\partial i_2^2}(\operatorname{tr}\boldsymbol{V}^{-1}\mathbf{1} - \boldsymbol{V}^{-1})\nabla i_2 + \frac{\partial^2 W}{\partial i_2\,\partial i_3}(\operatorname{tr}\boldsymbol{V}^{-1}\nabla i_3 - \boldsymbol{V}^{-1}\nabla i_3 + \nabla i_2) \\
&+ \frac{\partial^2 W}{\partial i_3^2}\nabla i_3 = \mathbf{0},
\end{aligned} \tag{1.9}$$

where ∇ is the gradient operator with respect to the current configuration.

We also note that the conditions that the strain energy and the stress vanish in the reference configuration are given by

$$\left.\begin{aligned} &W(3,3,1)=0,\\ &\frac{\partial W}{\partial i_1}+2\frac{\partial W}{\partial i_2}+\frac{\partial W}{\partial i_3}\bigg|_{i_1=i_2=3;\, i_3=1}=0.\end{aligned}\right\} \tag{1.10}$$

2. Spherical inflation and compaction

We assume a semi-inverse type of solution,

$$r=\hat{r}(R), \qquad \theta=\Theta, \qquad \phi=\Phi, \tag{2.1}$$

where (R, Θ, Φ) and (r, θ, ϕ) are the spherical coordinates of a particle before and after deformation. The deformation gradient tensor and the stretch tensor have physical components

$$\boldsymbol{F}=\boldsymbol{V}=\operatorname{diag}\left(\dot{r}, \frac{r}{R}, \frac{r}{R}\right), \tag{2.2}$$

where $\dot{r}=\mathrm{d}r/\mathrm{d}R>0$ and the principal invariants are

$$i_1=\dot{r}+2\frac{r}{R}, \qquad i_2=\frac{r^2}{R^2}+2\frac{r\dot{r}}{R}, \qquad i_3=\frac{\dot{r}r^2}{R^2}. \tag{2.3}$$

For spherical inflation (or compaction) as defined by (2.1), it is easy to show that the equations

$$\operatorname{div}(i_3^{-1}\boldsymbol{V})=\boldsymbol{0}, \qquad \operatorname{div}(\operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{1}-\boldsymbol{V}^{-1})=\boldsymbol{0} \tag{2.4}$$

are satisfied identically. Hence, for spherical inflation, the equations of equilibrium reduce to

$$\begin{aligned} &\frac{\partial^2 W}{\partial i_1^2} i_3^{-1}\boldsymbol{V}\nabla i_1+\frac{\partial^2 W}{\partial i_1\,\partial i_2}(\operatorname{tr}\boldsymbol{V}^{-1}\nabla i_1-\boldsymbol{V}^{-1}\nabla i_1+i_3^{-1}\boldsymbol{V}\nabla i_2)\\ &+\frac{\partial^2 W}{\partial i_1\,\partial i_3}(i_3^{-1}\boldsymbol{V}\nabla i_3+\nabla i_1)+\frac{\partial^2 W}{\partial i_2^2}(\operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{1}-\boldsymbol{V}^{-1})\nabla i_2\\ &+\frac{\partial^2 W}{\partial i_2\,\partial i_3}(\operatorname{tr}\boldsymbol{V}^{-1}\nabla i_3-\boldsymbol{V}^{-1}\nabla i_3+\nabla i_2)+\frac{\partial^2 W}{\partial i_3^2}\nabla i_3=\boldsymbol{0}, \end{aligned} \tag{2.5}$$

with $\boldsymbol{V}$ and i_1, i_2, and i_3 given by (2.2) and (2.3) respectively.

The structure of these equations suggests six forms of the strain energy function for which controllable deformations are possible. These strain energy functions have the form

$$\frac{\partial^2 W}{\partial i_k\,\partial i_l}\neq 0, \qquad \frac{\partial^2 W}{\partial i_m\,\partial i_n}=0 \qquad (m\neq k;\, n\neq l;\, k,l,m,n=1,2,3). \tag{2.6}$$

These six conditions are equivalent to the following forms of the strain energy function where f, g, and h are arbitrary functions of the appropriate invariant and

$c_1, \ldots, c_5$ are constants:

$$\left.\begin{array}{lll}
\text{(I)} & W=f(i_1)+c_1 i_2+c_2 i_3, & f''(i_1)\neq 0, \\
\text{(II)} & W=g(i_1)+c_1 i_1+c_2 i_3, & g''(i_2)\neq 0, \\
\text{(III)} & W=h(i_3)+c_1 i_1+c_2 i_2, & h''(i_3)\neq 0, \\
\text{(IV)} & W=c_1 i_1 i_2+c_2 i_1+c_3 i_2+c_4 i_3+c_5, & c_1\neq 0, \\
\text{(V)} & W=c_1 i_2 i_3+c_2 i_1+c_3 i_2+c_4 i_3+c_5, & c_1\neq 0, \\
\text{(VI)} & W=c_1 i_1 i_3+c_2 i_1+c_3 i_2+c_4 i_3+c_5, & c_1\neq 0.
\end{array}\right\} \quad (2.7)$$

Materials with strain energy functions of the forms (I)–(III) have recently been extensively studied by Carroll [5]. We will restrict our attention to the remaining 'mixed' forms of the strain energy function given in (IV), (V), and (VI). Following Carroll [5], these materials will be called materials of types IV, V, and VI.

There are a number of restrictions to be imposed on the strain energy functions as defined in (2.7). Before obtaining the radial deformation field for materials of types IV, V, and VI, we will next consider these restrictions.

3. Restrictions on the strain energy function

Requiring that the strain energy and the stress vanish in the reference configuration yields the following restrictions on application of (1.10).

Materials of type IV:

$$9c_1+3c_2+3c_3+c_4+c_5=0, \qquad 9c_1+c_2+2c_3+c_4=0. \tag{3.1}$$

Materials of type V:

$$3c_1+3c_2+3c_3+c_4+c_5=0, \qquad 5c_1+c_2+2c_3+c_4=0. \tag{3.2}$$

Materials of type VI:

$$3c_1+3c_2+3c_3+c_4+c_5=0, \qquad 4c_1+c_2+2c_3+c_4=0. \tag{3.3}$$

Now, on restriction to infinitesimal deformations, the strain energy functions of interest should reduce to the classical strain energy function of the linear theory. To ensure physically realistic behaviour within this range of deformation, we will assume positive shear and bulk moduli for each material. This results in the following restrictions on the material constants.

Materials of type IV:

$$c_2+c_3<0, \qquad 6c_1+c_2+c_3>0. \tag{3.4}$$

Materials of type V:

$$4c_1+c_2+c_3>0, \qquad 13c_1-2c_2-2c_3>0. \tag{3.5}$$

Materials of type VI:

$$c_1+c_2+c_3>0, \qquad c_1-2c_2-2c_3>0. \tag{3.6}$$

We now proceed to examine the equation for the radial deformation field describing spherical inflation for materials of types IV and V.

4. Materials of types IV and V

For materials of type IV, the equations of equilibrium (2.5) reduce to

$$2R^2 r\ddot{r} + R^2\dot{r}^2 + 2Rr\dot{r} - 3r^2 = 0. \tag{4.1}$$

We emphasize that this equation is independent of material constants and we note that the homogeneous deformation field $r = AR$, where A is constant, is a trivial solution to (4.1). We now seek nonhomogeneous deformation fields. To this end, we use the substitution introduced by Chung *et al.* [4] to solve a related problem. Let

$$t = R\dot{r}/r. \tag{4.2}$$

Equation (4.1) then becomes

$$2R\dot{t} + 3(t^2 - 1) = 0, \tag{4.3}$$

where $\dot{t} = dt/dR$. Following Chung *et al.* [4], we assume that

$$0 < t < 1. \tag{4.4}$$

Then an easy integration of (4.3) yields

$$R^3 = A\frac{1+t}{1-t}, \tag{4.5}$$

where A is a constant of integration.

Equations (4.2) and (4.3) together yield

$$\frac{1}{r}\frac{dr}{dt} = \frac{2t}{3(1-t^2)}. \tag{4.6}$$

On integrating, we obtain

$$r^3 = \frac{B}{1-t^2}, \tag{4.7}$$

where B is a constant of integration.

Elimination of t between (4.5) and (4.7) yields an explicit form of the radial deformation field,

$$r^3 = \frac{C}{R^3}(R^3 + D)^2, \tag{4.8}$$

where C and D are constants of integration.

For materials of type V, the equilibrium equations reduce to the form:

$$2R^2 r\ddot{r} + 3R^2\dot{r}^2 - 2Rr\dot{r} - r^2 = 0. \tag{4.9}$$

Proceeding exactly as before, we obtain the following solution to (4.9):

$$r^5 = \frac{C}{R}(R^3 + D)^2, \tag{4.10}$$

where C and D are arbitrary constants.

5. Materials of type VI

Armanni [7] considered spherical inflation for an elastic material with a strain energy function of the form

$$W = A(i_1 i_3 - 4i_3 + 1), \tag{5.1}$$

where A is a material constant. The corresponding deformation field describing spherical inflation was found to have the representation

$$r^2 = CR^2 + \frac{D}{R}, \qquad \theta = \Theta, \qquad \phi = \Phi, \tag{5.2}$$

where C and D are arbitrary constants.

The strain energy function (5.1) is easily seen to be a special case of the strain energy function of materials of type VI,

$$W = c_1 i_1 i_3 + c_2 i_1 + c_3 i_2 + c_4 i_3 + c_5, \tag{5.3}$$

where $c_1, \ldots, c_5$ satisfy (3.3). It can be verified immediately that (5.2) describes spherical inflation for the generalized Armanni material (5.3).

We now introduce the Carroll material having a strain energy function given by $(2.7)_2$:

$$W = g(i_2) + c_1 i_1 + c_2 i_3. \tag{5.4}$$

Carroll [5] has found that (5.2) also describes spherical inflation for these materials. We also note that John [10], using a result from linear algebra, has shown that the function

$$d(i_1, i_2, i_3) = (27i_3 + 2i_1^3 - 9i_1 i_2)^2 - 4(i_1^2 - 3i_2)^3 \tag{5.5}$$

vanishes for spherical inflation.

We conclude that (5.2) describes spherical inflation for the quite general strain energy function given by

$$W = c_1 i_1 + c_2 i_3 + c_3 i_1 i_3 + g(i_2) + a(i_1, i_2, i_3)d(i_1, i_2, i_3),$$

where $a(i_1, i_2, i_3)$ is an arbitrary sufficiently regular function. This strain energy function includes, of course, materials of type VI as a special case.

6. Qualitative features of inflation

In the recent review article by Beatty [8], the membrane inflation problem for the general Blatz–Ko material was considered. We will study the same problem for the three materials introduced in this paper and compare the results with those for the special Blatz–Ko material considered by Beatty [8] and more extensively by Chung *et al.* [4].

To avoid cumbersome algebra, we consider reduced forms of the strain energy functions of materials of types IV, V, and VI. Following Armanni [7], we consider the strain energy functions given below.

$$\left.\begin{aligned}&\text{Materials of type IV:} && W = c_1(i_1 i_2 - \tfrac{9}{2} i_2 + \tfrac{9}{2}),\\ &\text{Materials of type V:} && W = c_1(i_2 i_3 - 5 i_3 + 2),\\ &\text{Materials of type VI:} && W = c_1(i_1 i_3 - 4 i_3 + 1).\end{aligned}\right\} \tag{6.1}$$

Here, c_1 is a material constant. We will assume that $c_1 > 0$, and thus we can satisfy all the restrictions of Section 3.

For thin-walled spherical shells (or balloons) of compressible elastic material, the pressure–radius relation was obtained in a parametric form by Willson & Myers [11], as follows:

$$\frac{\partial W}{\partial \lambda_1} = 0, \qquad p = 2\epsilon (A/a)^2 \frac{\partial W}{\partial \lambda_2}, \tag{6.2}$$

where λ_1 and λ_2 are the principal stretches which in this case have the form

$$\lambda_1 = \alpha a/A, \qquad \lambda_2 = \lambda = a/A. \tag{6.3}$$

Here, p is the internal pressure, ϵ is the ratio of initial thickness to initial inner radius, A and a are the undeformed and deformed inner radii respectively, and α serves as a parameter.

Using (6.2) and (6.3), we can easily show that the pressure–radius relations for materials of types IV, V, and VI are as given below.

Materials of type IV:

$$\hat{p} = \frac{p}{4\epsilon\mu} = \tfrac{2}{3}(1 + 2\alpha)(1 - \alpha), \qquad \alpha = \frac{9 - 5\lambda}{4\lambda}, \tag{6.4}$$

where μ is the infinitesimal shear modulus. Requiring $\alpha > 0$, from $(6.3)_1$, restricts the range of λ to $1 \leqslant \lambda < \tfrac{9}{5}$. Therefore, at $\lambda = \tfrac{9}{5}$, we find that the determinant of the deformation gradient tensor vanishes. Thus, as we approach this critical finite value of stretch, the deformed shell-thickness approaches zero. Consequently the ballon bursts at $\lambda = \tfrac{9}{5}$.

Materials of type V:

$$\hat{p} = \frac{5\alpha(1 - \alpha)}{2(1 + 4\alpha)}, \qquad \alpha = \frac{5 - \lambda^2}{4\lambda^2}. \tag{6.5}$$

Here, $\alpha > 0$ implies that $1 \leqslant \lambda < \sqrt{5}$. In this case, the pressure–radius relation also has a turning point, and we see that the balloon bursts on returning to zero pressure.

Materials of type VI:

$$\hat{p} = \frac{4\alpha(1 - \alpha)}{1 + \alpha}, \qquad \alpha = \frac{2 - \lambda}{\lambda}. \tag{6.6}$$

Here, $\alpha > 0$ implies that $1 \leq \lambda < 2$, and again the balloon bursts when the internal pressure recovers the value zero.

As a comparison with other compressible materials, we will set down the pressure–radius relation for the Blatz–Ko material studied extensively by Chung *et al.* [4]. This strain energy function has the form

$$W = \mu[(i_2/i_3)^2 - 2i_1/i_3 + 2i_3 - 5], \tag{6.7}$$

where μ is again the infinitesimal shear modulus. It is easily shown that the pressure–radius relation is given by

$$\hat{p} = \alpha(1 - \alpha^2), \qquad \alpha^3 = 1/\lambda^5. \tag{6.8}$$

Thus, $\alpha > 0$ for $1 \leq \lambda < \infty$. It is easily shown that $\hat{p}(\lambda)$ has a single turning point and that $\hat{p} \to 0$ as $\lambda \to \infty$.

In Fig. 1 we plot the above nondimensionalized pressure–radius relations for each of the three materials introduced in this paper and for the Blatz–Ko material. All these materials have a common qualitative feature: each plot has only one turning point. The differences are more striking however: the Blatz–Ko material does not admit a pressure at which the balloon bursts; materials of type

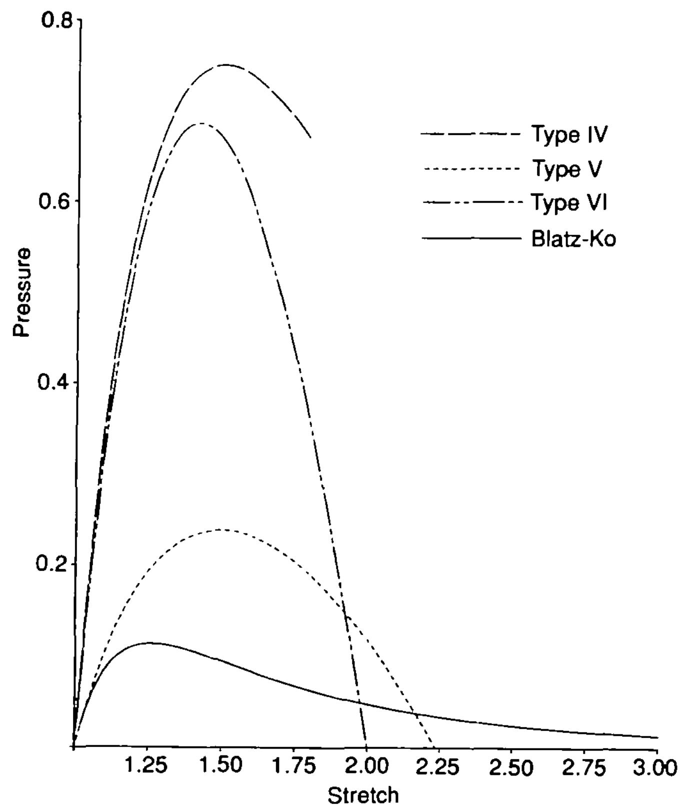

FIG. 1. Pressure–radius relations.

IV burst at a nonzero pressure and finite stretch, whereas materials of types V and VI will burst at a finite stretch when the value of zero pressure is reached on the strain-softening portion of the curve.

7. Spherical eversion

In Armanni's significant paper [7], both spherical inflation and eversion were considered for the Armanni material defined by (5.1). Motivated by this, we now consider spherical eversion for materials of types IV, V, and VI.

A deformation with spherical coordinate representation

$$r = \hat{r}(R), \qquad \theta = \pi - \Theta, \qquad \phi = \Phi, \tag{7.1}$$

with $dr/dR < 0$, describes eversion of a sphere. The deformation gradient tensor has physical components

$$\boldsymbol{F} = \mathrm{diag}\left(\frac{dr}{dR}, -\frac{r}{R}, \frac{r}{R}\right), \tag{7.2}$$

and a polar decomposition of $\boldsymbol{F}$ yields the stretch and rotation tensors in physical components, as follows:

$$\boldsymbol{R} = \mathrm{diag}\,(1, -1, 1), \tag{7.3}$$

$$\boldsymbol{V} = \mathrm{diag}\left(-\frac{dr}{dR}, \frac{r}{R}, \frac{r}{R}\right). \tag{7.4}$$

We first note that, for deformations of the form (7.1), we have

$$\mathrm{div}\,(i_3^{-1}\boldsymbol{V}) = \frac{4R}{\dot{r}r^2}, \qquad \mathrm{div}\,(\mathrm{tr}\,\boldsymbol{V}^{-1}\mathbf{1} - \boldsymbol{V}^{-1}) = \frac{4}{\dot{r}r}, \tag{7.5}$$

and consequently the equation for the determination of the radial deformation field will *not* be independent of material constants for each of the materials of types IV, V, and VI. In fact, the equations of equilibrium reduce to the following forms.

Materials of type IV:

$$c_2 2R^2 + c_3 2rR + c_1(3r^2 - 2Rr\dot{r} - \dot{r}^2R^2 - 2R^2r\ddot{r}) = 0, \tag{7.6}$$

Materials of type V:

$$c_2 2R^4 + c_3 2rR^3 + c_1(2r^3\dot{r}R - 2r^3R^2\ddot{r} - 3r^2R^2\dot{r}^2 - r^4) = 0, \tag{7.7}$$

Materials of type VI:

$$c_2 2R^3 + c_3 2rR^2 - c_1(r^2R^2\ddot{r} + rR^2\dot{r}^2 + r^3) = 0. \tag{7.8}$$

Closed-form solutions to (7.6), (7.7), and (7.8) are not immediately obvious. We will use the substitution introduced by Armanni [7]:

$$t = -R\dot{r}/r. \tag{7.9}$$

The above equations can then be written as follows.

Materials of type IV:

$$c_2 2R^2 + c_3 2rR + c_1 r^2(3 - 3t^2 + 2R\dot{t}) = 0, \tag{7.10}$$

Materials of type V:

$$c_2 2R^4 + c_3 2rR^3 + c_1 r^4(2R\dot{t} - 4t - 5t^2 - 1) = 0, \tag{7.11}$$

Materials of type VI:

$$c_2 2R^3 + c_3 2rR^2 + c_1 r^3(R\dot{t} - t - 2t^2 - 1) = 0. \tag{7.12}$$

An examination of (7.10), (7.11), and (7.12) reveals that (7.9) reduces the equations of equilibrium to an ODE of the first order in t for all the materials of interest when

$$c_2 = c_3 = 0, \qquad c_1 > 0. \tag{7.13}$$

However, we see that these conditions are not admissible for materials of type IV from $(3.4)_1$. We proceed now assuming (7.13) to hold for materials of types V and VI and obtain the following closed-form solutions describing the radial displacement field for spherical eversion for these materials.

Materials of type V:

$$r^5 = \frac{c_1 R^2}{1 + \tan^2(\frac{1}{2}\ln Rc_2)}, \tag{7.14}$$

Materials of type VI:

$$r^4 = \frac{c_1 R}{1 + \tan^2(\frac{1}{2}\sqrt{7}\ln Rc_2)}, \tag{7.15}$$

where c_1 and c_2 are constants of integration. Equation (7.15) is essentially the same equation as that obtained by Armanni [7] except that a mistake was made in the integration in [7]. The solution (7.14) appears to be new.

8. Cylindrical inflation

Deformations having cylindrical coordinate representation

$$r = \hat{r}(R), \qquad \theta = \Theta, \qquad z = \gamma Z, \tag{8.1}$$

with $\mathrm{d}r/\mathrm{d}R > 0$ and $\gamma > 0$, describe radial expansion or compaction of hollow cylinders with axial stretch γ. The deformation gradient tensor and the stretch tensor have physical components

$$\boldsymbol{F} = \boldsymbol{V} = \operatorname{diag}\left(\frac{\mathrm{d}r}{\mathrm{d}R}, \frac{r}{R}, \gamma\right), \tag{8.2}$$

and the principal invariants are

$$i_1 = \frac{\mathrm{d}r}{\mathrm{d}R} + \frac{r}{R} + \gamma, \quad i_2 = \left(\frac{r}{R} + \gamma\right)\frac{\mathrm{d}r}{\mathrm{d}R} + \gamma\frac{r}{R}, \quad i_3 = \gamma\frac{r}{R}\frac{\mathrm{d}r}{\mathrm{d}R}. \tag{8.3}$$

Deformations having a coordinate representation (8.1) identically satisfy (2.4), and the equations of equilibrium therefore have the form given by (2.5) where now V is given by (8.2), and i_1, i_2, and i_3 are given by (8.3). We then find that the radial deformation field describing cylindrical inflation (and compression) is given by the equations below for materials of types IV, V, and VI.

Materials of type IV:

$$2\ddot{r}\left(\frac{R}{r}+\frac{1}{\gamma}\right)+\frac{\dot{r}}{r}\left(1+\frac{\dot{r}}{\gamma}\right)-\frac{1}{R}\left(2+\frac{r}{\gamma R}\right)=0, \tag{8.4}$$

Materials of type V:

$$2\ddot{r}\left(\frac{r}{R}+\gamma\right)+\frac{2\dot{r}^2}{R}-\frac{2r\dot{r}}{R^2}+\frac{\gamma\dot{r}^2}{r}-\frac{\gamma r}{R^2}=0, \tag{8.5}$$

Materials of type VI:

$$2\ddot{r}+\frac{\dot{r}^2}{r}-\frac{r}{R^2}=0. \tag{8.6}$$

In a recent comprehensive treatment of the substitution (4.2), i.e. $t=R\dot{r}/r$, Horgan [12] has shown that, for the materials considered by him, this substitution reduces the equation for the radial displacement field for *both* spherical and cylindrical inflation to an ODE of the first order. For the materials considered in this paper, we have already shown how (4.2) similarly reduces the equation describing spherical inflation and allows a closed-form solution to be obtained. Surprisingly (4.2) does not reduce the equations describing cylindrical inflation to an ODE of the first order for materials of types IV and V, and equations (8.4) and (8.5) do not have obvious closed-form solutions. Use of (4.2), however, does reduce (8.6) to a first order ODE, and the form of the radial deformation field describing cylindrical inflation for materials of type VI is easily shown to be

$$r^2=\frac{C}{R}(R^2+D), \tag{8.7}$$

where C and D are constants of integration.

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