Elasticity HW4

A rigid body consists of eight particles, each of mass m, held together by light rods. In a certain coordinate frame the particles are at positions

$$\pm a(3,1,-1),\quad \pm a(1,-1,3),\quad \pm a(1,3,-1),\quad \pm a(-1,1,3)$$

If the body rotates about an axis through the origin, the angular velocity and angular momentum vectors may parallel. Please find their ratios. Also find the three eigenvectors and plot the Mohr's circle.

Sol：

$$I_{ij} = \int (r^2\delta_{ij} - x_i x_j)dm$$

$$I_{11} = \sum (y^2 + z^2)m$$
$$= \{[a^2 + (-a)^2] + [(-a)^2 + a^2] + [(-a)^2 + (3a)^2] + [(a)^2 + (-3a)^2] + [(3a)^2 + (-a)^2] +$$
$$[(-3a)^2 + a^2] + [a^2 + (3a)^2] + [(-a)^2 + (-3a)^2]\}m$$
$$= 64ma^2$$

$$I_{12} = -\sum (xy)m$$
$$= (3a^2 + 3a^2 - a^2 - a^2 + 3a^2 + 3a^2 - a^2 - a^2)m$$
$$= -8ma^2$$

$$I_{13} = -\sum (xz)m$$
$$= (-3a^2 - 3a^2 + 3a^2 + 3a^2 - a^2 - a^2 - 3a^2 - 3a^2)m$$
$$= 8ma^2$$

$$I_{22} = \sum (x^2 + z^2)m$$
$$= \{[(3a)^2 + (-a)^2] + [(-3a)^2 + a^2] + [a^2 + (3a)^2] + [(-a)^2 + (-3a)^2] + [a^2 + (-a)^2] +$$
$$[(-a)^2 + a^2] + [(-a)^2 + (3a)^2] + [a^2 + (-3a)^2]\}m$$
$$= 64ma^2$$

$$I_{23} = -\sum (yz)m$$
$$= (-a^2 - a^2 - 3a^2 - 3a^2 - 3a^2 - 3a^2 + 3a^2 + 3a^2)m$$
$$= 8ma^2$$

$$I_{33} = \sum (x^2 + y^2)m$$
$$= \{[(3a)^2 + a^2] + [(-3a)^2 + (-a)^2] + [a^2 + (-a)^2] + [(-a)^2 + a^2] + [a^2 + (3a)^2] +$$
$$[(-a)^2 + (-3a)^2] + [(-a)^2 + a^2] + [a^2 + (-a)^2]\}m$$
$$= 48ma^2$$

$$[I] = \begin{bmatrix} 64ma^2 & -8ma^2 & 8ma^2 \\ & 64ma^2 & 8ma^2 \\ sym & & 48ma^2 \end{bmatrix}$$

$$[I]\underset{\sim}{\omega} = \lambda\underset{\sim}{\omega}$$

$$\begin{bmatrix} 64ma^2-\lambda & -8ma^2 & 8ma^2 \\ -8ma^2 & 64ma^2-\lambda & 8ma^2 \\ 8ma^2 & 8ma^2 & 48ma^2-\lambda \end{bmatrix}\underset{\sim}{\omega} = 0$$

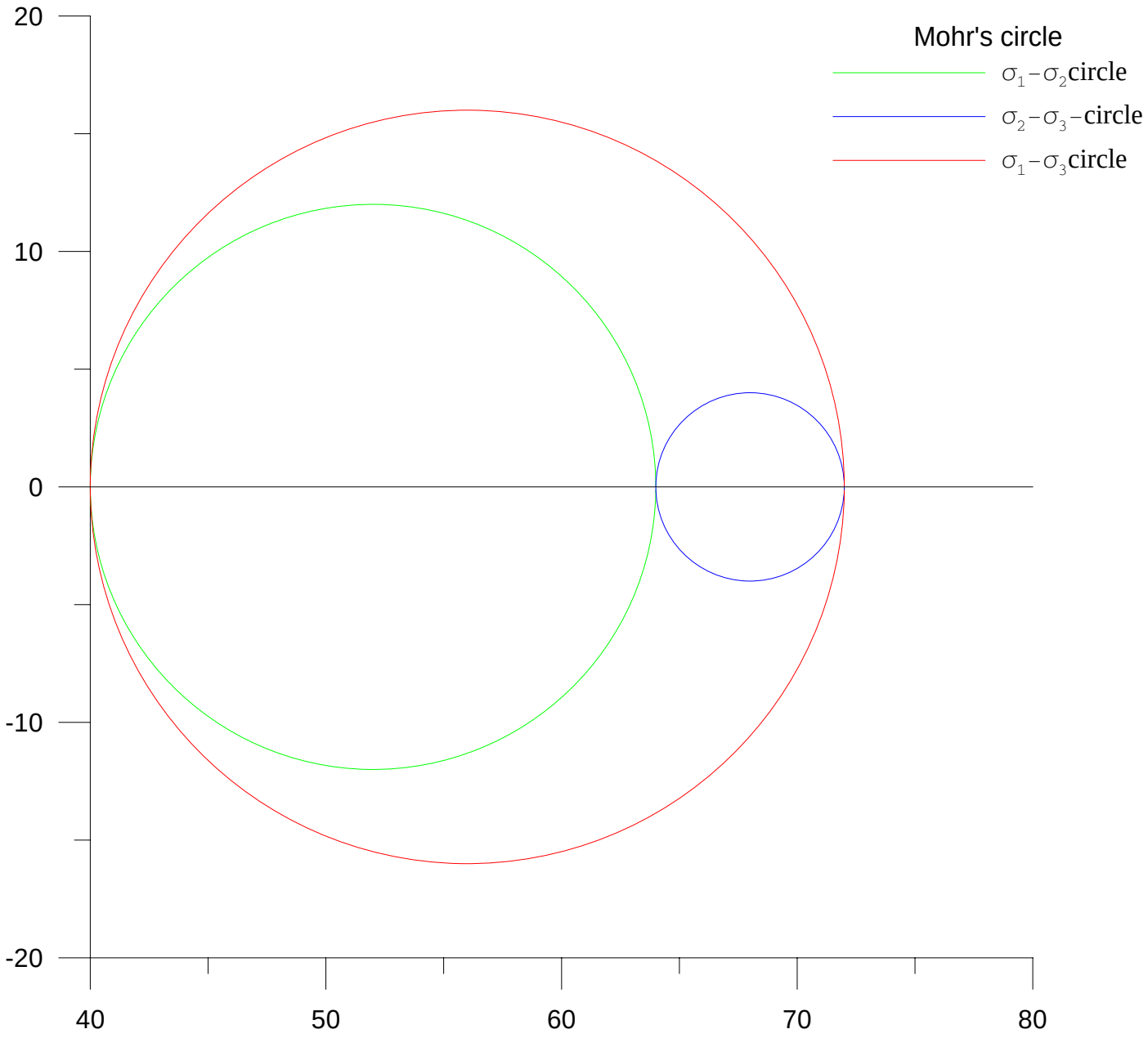
$$\det\left[\begin{bmatrix} 64ma^2-\lambda & -8ma^2 & 8ma^2 \\ -8ma^2 & 64ma^2-\lambda & 8ma^2 \\ 8ma^2 & 8ma^2 & 48ma^2-\lambda \end{bmatrix}\right] = 0$$

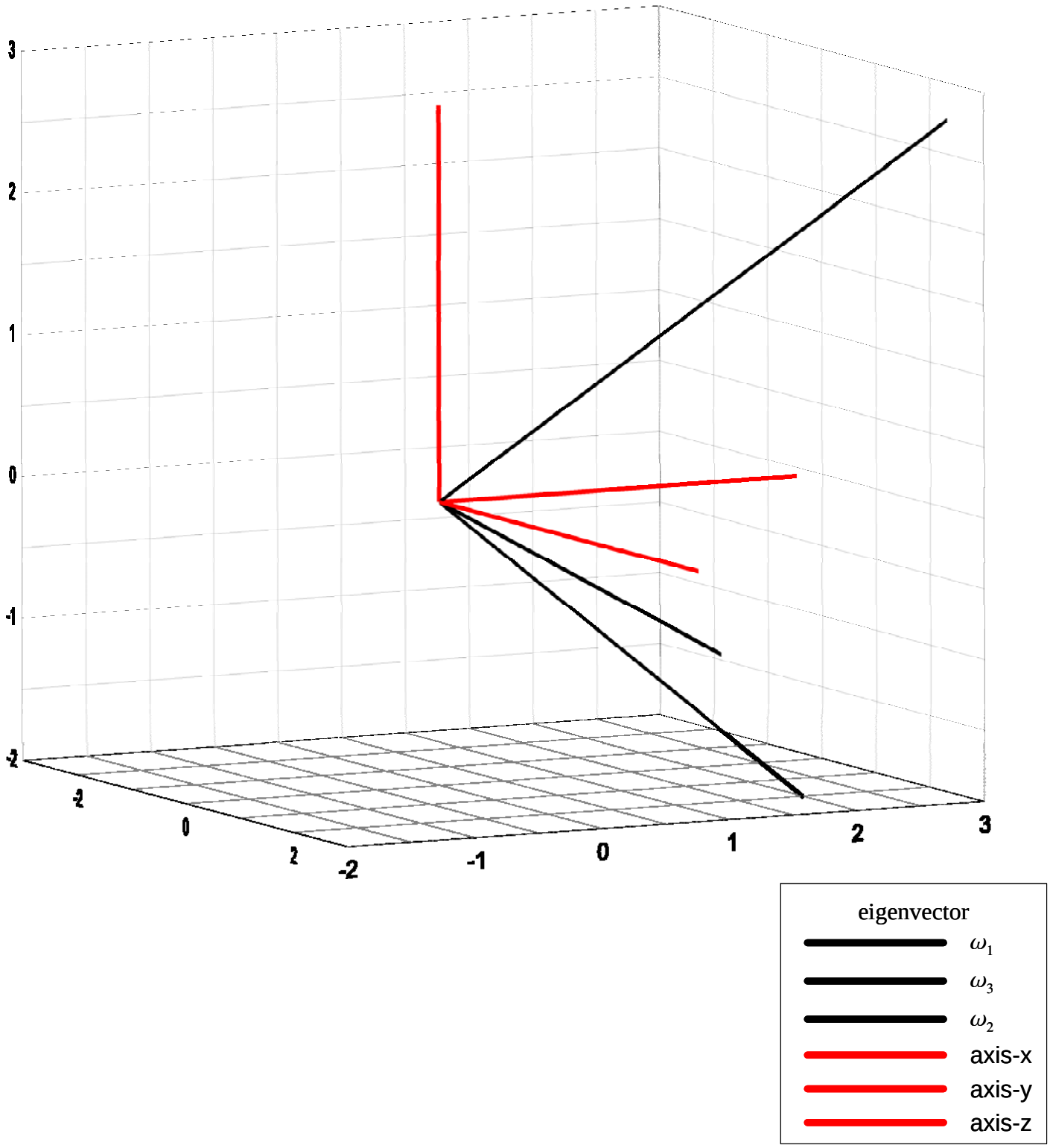
$$\lambda = 72ma^2, 64ma^2, 40ma^2$$

let $\lambda_1 = 40ma^2, \quad \lambda_2 = 64ma^2, \quad \lambda_3 = 72ma^2$

so

$$\underset{\sim}{\omega}_1 = \{-1,-1,2\}$$
$$\underset{\sim}{\omega}_2 = \{1,1,1\}$$
$$\underset{\sim}{\omega}_3 = \{-1,1,0\}$$


Mohr's circle
$\sigma_1-\sigma_2$circle
$\sigma_2-\sigma_3-$circle
$\sigma_1-\sigma_3$circle
20
10
0
-10
-20
40
50
60
70
80


3
2
1
0
-1
-2
-2
0
2
-2
-1
0
1
2
3
eigenvector
ω_1
ω_3
ω_2
axis-x
axis-y
axis-z