Given a matrix

$$F = \begin{bmatrix} 1 & 2/\sqrt{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find

(1) $F^T F, \sqrt{F^T F}, FF^T \text{ and } \sqrt{FF^T}$

(2) IF F=RU, find R and U.

(3) F=VR, find V.

(4) F=$\Phi\,\Sigma\,\Psi^T$**, find** Φ**,** Ψ **and** Σ**.**

(5) Verify R=$\Phi\,\Psi^T$**.**

Sol:

(1)

$$F^T F = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ \frac{2}{\sqrt{3}} & \frac{7}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$FF^T = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & \frac{2}{\sqrt{3}} & 0 \\ \frac{2}{\sqrt{3}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $F^T F$ **eigenvalues and eigenvectors**

$\lambda_1 = 3 \quad \psi_1 = \{\frac{1}{\sqrt{3}}, 1, 0\}$

$\lambda_2 = 1 \quad \psi_2 = \{0, 0, 1\}$

$\lambda_3 = \frac{1}{3} \quad \psi_3 = \{-\sqrt{3}, 1, 0\}$

Let $\Psi = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}, \quad \Psi^{-1} = \Psi^{T} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$$F^{T}F = \Psi \sum \Psi^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\sqrt{F^{T}F} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find FF^{T} eigenvalues and eigenvectors

$\lambda_1 = 3 \quad \phi_1 = \{\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\}$

$\lambda_2 = 1 \quad \phi_2 = \{0, 0, 1\}$

$\lambda_3 = \frac{1}{3} \quad \phi_3 = \{\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0\}$

Let $\Phi = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \end{bmatrix}, \quad \Phi^{-1} = \Phi^{T} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$

$$FF^{T} = \Phi \sum \Phi^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

$$\sqrt{FF^T} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2\sqrt{3}} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2)

$$F = RU$$

$$F^T F = U^T R^T R U = U^T U = U^2$$

$$U = \sqrt{F^T F} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

$$F = VR$$

$$FF^T = VRR^T V^T = V^2$$

$$V = \sqrt{FF^T} = \begin{bmatrix} \frac{5}{2\sqrt{3}} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4)

$$F = \Phi \Sigma \Psi^T$$

$$FF^T = \Phi \Sigma \Psi^T \Psi \Sigma^T \Phi^T = \Phi \Sigma^2 \Phi^T, \quad F^T F = \Psi \Sigma^T \Phi^T \Phi \Sigma \Psi^T = \Psi \Sigma^2 \Psi^T$$

$$\Phi = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

(5)

$$F = RU = R(\Psi \Sigma \Psi^T) = \Phi \Sigma \Psi^T$$

$$R\Psi = \Phi$$

$$R = \Phi \Psi^T$$