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On the Application of a Baecklund Transformation to Linear Isotropic Elasticity

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Baecklund transformations have been employed in gas-dynamics to reduce the hodograph equations to appropriate canonical forms in subsonic, transonic and supersonic flows; thus, for example, the important Kármán–Tsien approximation may be generated as a consequence of a simple Baecklund transformation of the hodograph system. Here, it is shown that Weinstein's correspondence principle in generalized axially symmetric potential theory emerges as a particular member of a class of Baecklund transformations of the Stokes–Beltrami equations. An iterated form of the correspondence principle may be used to obtain solutions to certain boundary-value problems involving axially-symmetric deformations of an incompressible isotropic linear elastic material. Such solutions assume an added importance in the light of recent work by Selvadurai & Spencer, where the first order theory serves as the basis for solutions in second order incompressible finite elasticity.

1. Introduction

THE PRESENT work falls naturally into two parts. Firstly, in Section 2, we introduce Baecklund transformations which transform the Stokes–Beltrami system

$$\frac{\partial \phi}{\partial y} = -y^{-p}\frac{\partial \psi}{\partial x}, \qquad \frac{\partial \psi}{\partial y} = y^{p}\frac{\partial \phi}{\partial x},$$

to the associated system

$$\frac{\partial \phi'}{\partial y'} = -y'^{-q}\frac{\partial \psi'}{\partial x'}, \qquad \frac{\partial \psi'}{\partial y'} = y'^{q}\frac{\partial \phi'}{\partial x'}.$$

Secondly, in Section 3, one of the Baecklund transformations is used to facilitate the solution of a class of axially symmetric boundary value problems for an isotropic elastic solid.

Essentially, this is an extension of previous work on generalized axially symmetric potential theory by various authors; notably Weinstein (1953), Weiss & Payne (1954), Payne & Pell (1960), Pell & Payne (1960*a*, *b*) and Burns (1970). These authors employed a particular transformation of the Stokes–Beltrami system to simplify, and hence solve, various boundary value problems in fluid dynamics and also problems involving the torsion of axially symmetric shafts in elasticity. In this paper we show that the transformation employed by these authors is but one of a class of Baecklund transformations for the Stokes–Beltrami system. Also, we show that the elastic problems considered by these authors are a special case of a wider class of elastic problems which can be solved by employing transformations of the Stokes–Beltrami system.

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2. The Baecklund Transformation

Transformations

$$u \to u^*; \qquad \xi_j \to \xi_j^* \qquad j = 1, 2,$$

defined by relations of the form

$$\beta_i(\xi_1, \xi_2, u, u_{\xi_1}, u_{\xi_2}; \xi_1^*, \xi_2^*, u^*, u^*_{\xi_1^*}, u^*_{\xi_2^*}) = 0, \qquad i = 1, 2, 3, 4 \tag{2.1}$$

$$\left(u_{\xi_j} \equiv \frac{\partial u}{\partial \xi_j}, \qquad j = 1, 2\right)$$

were first introduced by Baecklund (1882) in connection with the transformation of surfaces in (ξ_1, ξ_2, u) space to surfaces in (ξ_1^*, ξ_2^*, u^*) space. Their role in the transformation of pseudo-spherical surfaces leaving the total mean curvature K invariant is discussed by Eisenhart (1960: 284). More generally, "$(m, 2)$—Baecklund transformations" of the type

$$\begin{aligned}\beta_i(\xi_1, \xi_2, u_1, \ldots, u_m, u_{1,\xi_1}, \ldots u_{m,\xi_1}, u_{1,\xi_2} \ldots u_{m,\xi_2};\\ \xi_1^*, \xi_2^*, u_1^*, \ldots u_m^*, u^*_{1,\xi_1^*}, \ldots u^*_{m,\xi_1^*}, u^*_{1,\xi_2^*}, \ldots u^*_{m,\xi_2^*}) = 0,\end{aligned} \tag{2.2}$$

$$i = 1, \ldots, 2m+2$$

$$\left(u_{k,\xi_j} \equiv \frac{\partial u_k}{\partial \xi_j} \qquad k = 1, \ldots, m; \qquad j = 1, 2\right)$$

may be applied to general linear first order partial differential equations for m functions u_k, $k = 1, \ldots, m$ (in the two independent variables ξ_1, ξ_2) of the form

$$\sum_{k=1}^{m} \alpha_{ik} u_{k,\xi_2} + \sum_{k=1}^{m} \beta_{ik} u_{k,\xi_1} + \sum_{k=1}^{m} \gamma_{ik} u_k + \delta_i = 0, \qquad (i = 1, \ldots, m) \tag{2.3}$$

where $\alpha_{ik}, \beta_{ik}, \gamma_{ik}, \delta_i$ are functions of the ξ_j. In particular, for $m = 2$ (Rogers, in press) such transformations have importance when applied to hodograph systems in gas-dynamics and magneto-gas-dynamics (Loewner, 1950, Power, Rogers & Osborn, 1969, Rogers, 1970). Other applications occur in dislocation theory (Seeger, 1955), the study of long Josephson junctions (Scott, 1970), and the propagation of very short optical pulses through a resonant laser medium (Lamb, 1971). Moreover, the important recent discovery that the Korteweg de-Vries equation possesses conservation laws in addition to those of field energy and field momentum is derived by implicit use of Baecklund transformations. Finally, certain important results derived by Cekirge & Varley (1973) on large amplitude disturbances in bounded media may be derived in a remarkable manner as a consequence of Baecklund transformations.

It is now shown how a correspondence principle derived by Weinstein and applied by him to torsion problems in elasticity may be generated as a Baecklund transformation on a Stokes–Beltrami system. Thus, Rogers & Kingston (1973) recently investigated Baecklund transformations of the type (2.2) with $m = 2$ defined by

$$\begin{aligned}\mathbf{\Omega}'_x &= \bar{\mathbf{A}}\mathbf{\Omega}_x + \bar{\mathbf{B}}\mathbf{\Omega} + \bar{\mathbf{C}}\mathbf{\Omega}', \qquad |\bar{\mathbf{A}}| \neq 0,\\ \mathbf{\Omega}'_y &= \mathbf{A}\mathbf{\Omega}_y + \mathbf{B}\mathbf{\Omega} + \mathbf{C}\mathbf{\Omega}', \qquad |\mathbf{A}| \neq 0,\\ x' &= x, \qquad y' = y, \qquad \mathbf{\Omega} = \begin{pmatrix}\phi\\ \psi\end{pmatrix}, \qquad \mathbf{\Omega}' = \begin{pmatrix}\phi'\\ \psi'\end{pmatrix},\end{aligned} \tag{2.4}$$

which transform a Stokes–Beltrami system

$$\mathbf{\Omega}_y = \mathbf{\Lambda}\mathbf{\Omega}_x, \qquad \mathbf{\Lambda} = \begin{bmatrix} 0 & -y^{-p} \\ y^p & 0 \end{bmatrix}, \tag{2.5}$$

to an associated system

$$\mathbf{\Omega}'_{y'} = \mathbf{\Lambda}'\mathbf{\Omega}'_{x'}, \mathbf{\Lambda}' = \begin{bmatrix} 0 & -y'^{-q} \\ y'^q & 0 \end{bmatrix}. \tag{2.6}$$

In the above, $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are 2×2 matrix functions of x and y. Both invariance and reducibility properties were exhibited and, in particular, linking Baecklund transformations were established when $q = p+2$, $p \neq -1$ and

$$\mathbf{A} = \bar{\mathbf{A}} = \begin{bmatrix} 2(ax+b) & y^{-p-1}[a(x^2-y^2)+2(bx+d)] \\ -y^{p+1}[a(x^2-y^2)+2(bx+d)] & 2y^2(ax+b) \end{bmatrix},$$

$$\bar{\mathbf{B}} = \begin{bmatrix} a(p+2) & 2y^{-p-1}(ax+b) \\ 0 & ay^2+(p+1)[ax^2+2(bx+d)] \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & -ay^{-p}-(p+1)y^{-p-2}[ax^2+2(bx+d)] \\ a(p+2)y^{p+2} & 2y(ax+b) \end{bmatrix},$$

$$\bar{\mathbf{C}} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & -cy^{-p-2} \\ cy^{p+2} & 0 \end{bmatrix},$$

where a, b, c and d are arbitrary constants, except that they are assumed not to all vanish simultaneously. If we now set

$$a = b = c = 0, \qquad d \neq 0,$$

the system (2.4) gives, for ϕ',

$$\phi'_x = 2dy^{-p-1}\psi_x, \qquad \phi'_y = 2dy^{-p-1}\psi_y - 2d(p+1)y^{-p-2}\psi,$$

whence

$$\phi' = 2dy^{-p-1}\psi, \qquad p \neq -1, \tag{2.7}$$

or, in the notation of Weinstein,

$$\psi\{p\} = Cy^{p+1}\phi\{p+2\}, \qquad p \neq -1 \quad (C = 1/(2d)). \tag{2.8}$$

Hence, Weinstein's correspondence principle may be regarded as a Baecklund transformation of the Stokes–Beltrami equations. Associated with a specific $\psi(p)$ there is a function $\phi(p+2)$ in two more dimensions, determined up to the multiplicative constant C. Moreover, the above approach provides a four-parameter class of correspondence principles embodying that due to Weinstein as a simple particular case. However, it is not our intention to exploit this interesting fact here, but rather to give a new application of an iterated form of Weinstein's Correspondence Principle. Thus, if solutions of the iterated equation of axially symmetric potential theory

$$L_k^n(f) = 0, \qquad (L_k(f) \equiv f_{xx}+f_{yy}+ky^{-1}f_y, L_k^n=(L_k)^n) \tag{2.9}$$

are denoted by $f_k^{(n)}$, then Weinstein's principle may be written in the form

$$f_k^{(1)} \leftrightarrow y^{1-k}f_{2-k}^{(1)} \tag{2.10}$$

and induction readily establishes the result due to Burns (1967) that

$$f_k^{(n)} \leftrightarrow y^{1-k}f_{2-k}^{(n)}. \tag{2.11}$$

Burns (1970) applied this result in the case $n = 2$ to systematize the study of problems involving Stokes flow of a viscous fluid past such bodies as a spindle, lens or torus

The work was compared with that of Pell & Payne (1960*a*, *b*). In the next section, it is shown that the same result may be used to solve certain small deformation problems of an incompressible elastic material with a rigid adhering inclusion.

3. The Basic Equations

Cylindrical polar coordinates (r, θ, z) are adopted and small deformations independent of θ are investigated for an isotropic elastic solid. Consequently, the displacements u_r, u_θ, u_z are related to the strains ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz}, by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{3.1}$$

and the dilatation Δ is given by

$$\Delta = \frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}, \tag{3.2}$$

and for an incompressible elastic material, it is required that $\Delta = 0$. In the absence of body forces, the equilibrium equations reduce to

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r}+\frac{\partial \sigma_{rz}}{\partial z}+(\sigma_{rr}-\sigma_{\theta\theta})r^{-1} &= 0, \\ \frac{\partial \sigma_{r\theta}}{\partial r}+\frac{\partial \sigma_{\theta z}}{\partial z}+2\sigma_{r\theta}r^{-1} &= 0, \\ \frac{\partial \sigma_{rz}}{\partial r}+\frac{\partial \sigma_{zz}}{\partial z}+\sigma_{rz}r^{-1} &= 0. \end{aligned} \tag{3.3}$$

In order to satisfy identically the incompressibility condition $\Delta = 0$ the displacement function $\Psi_1(r, z)$ is introduced according to the relations

$$u_r = r^{-1}\frac{\partial \Psi_1}{\partial z}, \qquad u_z = -r^{-1}\frac{\partial \Psi_1}{\partial r}. \tag{3.4}$$

Now, the components of stress are given by

$$\begin{aligned} \sigma_{rr} &= -p+2\mu\varepsilon_{rr} = -p+2\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi_1}{\partial z}\right), \\ \sigma_{\theta\theta} &= -p+2\mu\varepsilon_{\theta\theta} = -p+2\mu r^{-2}\frac{\partial \Psi_1}{\partial z}, \\ \sigma_{zz} &= -p+2\mu\varepsilon_{zz} = -p+2\mu r^{-1}\frac{\partial^2 \Psi_1}{\partial z^2}, \\ \sigma_{r\theta} &= 2\mu\varepsilon_{r\theta} = \mu\left[\frac{\partial u_\theta}{\partial r}-r^{-1}u_\theta\right], \\ \sigma_{rz} &= 2\mu\varepsilon_{rz} = \mu\left[\frac{1}{r}\frac{\partial^2 \Psi_1}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi_1}{\partial r}\right)\right], \\ \sigma_{\theta z} &= 2\mu\varepsilon_{\theta z} = \mu\frac{\partial u_\theta}{\partial z}, \end{aligned} \tag{3.5}$$

where p is the isotropic pressure. On substitution of the stress components into the equilibrium equations (3.3), it is seen that

$$\mu\left\{r^{-1}\frac{\partial}{\partial z}(E^2\Psi_1)\right\}-\frac{\partial p}{\partial r} = 0, \tag{3.6}$$

$$\mu\left\{r^{-1}\frac{\partial}{\partial r}(E^2\Psi_1)\right\}+\frac{\partial p}{\partial z} = 0, \tag{3.7}$$

$$\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial z^2} = 0, \tag{3.8}$$

$$E^2 \equiv \frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \equiv L^1_{-1}. \tag{3.9}$$

Elimination of p and Ψ_1 in turn from (3.6) and (3.7) yields

$$E^4\Psi_1 \equiv L^2_{-1}\Psi_1 = 0, \tag{3.10}$$

together with

$$\nabla^2 p \equiv \frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial z^2} \equiv L^1_1 p = 0. \tag{3.11}$$

Moreover, setting

$$u_\theta = \Psi_2/r, \tag{3.12}$$

equation (3.8) shows that

$$L^1_{-1}\Psi_2 = 0. \tag{3.13}$$

It is observed that the deformation under consideration is composed of two independent parts. Firstly, the rotational displacement (3.12) is governed by (3.13) and gives rise to the components of stress $\sigma_{r\theta}$ and $\sigma_{\theta z}$; moreover, it does not contribute to the dilatation and consequently is unrelated to the compressibility of the material. Setting $p = 0$, Ψ_1 = constant, the torsion equations investigated by Weinstein (1953) and Weiss & Payne (1954) are obtained. The only distinction between the compressible and incompressible deformations in their analysis is the presence of the harmonic scalar function in the case of an incompressible material. Secondly, that part of the displacement with components u_r and u_z is governed by (3.10) and gives rise to the components of stress σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{rz}. It only applies to the deformations of incompressible materials and consequently the components of stress σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} include a contribution due to the scalar pressure function p.

In the work of Weinstein on the torsion of axially-symmetric shafts with cavities, any surface Ψ_2 = constant is stress-free and can be taken as the profile of a free surface bounding the shaft. In the subsequent section, deformations with Ψ_1 non-constant are investigated.

4. Deformation of an Incompressible Elastic Material with a Rigid Inclusion

Small deformations of an incompressible elastic material containing a rigid inclusion are considered; the inclusion is assumed to adhere to the surrounding elastic material. Only deformations with zero torsional component are discussed, so that Ψ_2 vanishes; it is apparent that the displacement field is given by the pair of relations (3.4) where Ψ_1 satisfies (3.10).

It is required to determine a function $\Psi_1(r, z)$ which satisfies (3.10) in the region $r \geqslant 0$ exterior to a boundary which can be of two types (see Burns, 1970) namely

(a) an arc l joining two points on the z-axis, but otherwise lying above it, or

(b) a closed curve l' lying above the z-axis.

In fact, the analysis may be extended to cover the more general case when there are several curves l or l' (Payne & Pell, 1960).

In the present context, since the rigid inclusion is assumed to adhere to the material,

$$\frac{\partial \Psi_1}{\partial z} = \frac{\partial \Psi_1}{\partial r} = 0 \qquad \text{on } l \text{ or } l',$$

where l or l' define the boundary of the inclusion. Provided l and l' have their tangents parallel to the z-axis only at isolated points, it is apparent that the above boundary conditions may be replaced by

$$\Psi_1 = \Psi_0, \qquad \frac{\partial \Psi_1}{\partial z} = 0 \qquad \text{on } l \text{ or } l',$$

where Ψ_0 is a constant, which is zero in case (a) and non-zero in case (b).

Finally, it is assumed that the displacement component u_z satisfies the uniformity condition

$$\lim_{|R| \to \infty} u_z = -U,$$

where U is a constant and $R^2 = r^2 + z^2$; this condition converts to the requirement

$$\lim_{|R| \to \infty} \Psi_1(r, z) = \tfrac{1}{2} U r^2,$$

on Ψ_1. The problem for the determination of $\Psi_1(r, z)$ is now reduced to the form investigated by Burns (1970) in connection with Stokes' flow of a viscous fluid; as a consequence, each of the solutions presented in that paper generates the solution to an associated inclusion problem in elasticity.

As a simple illustration, consider the case of a rigid spherical inclusion occupying the region $|R| < a$. The function Ψ_1 is conveniently represented in the form

$$\Psi_1(r, z) = \tfrac{1}{2} U r^2 - \Psi_1^*(r, z), \tag{4.1}$$

where

$$L_{-1}^2 \Psi_1^* = 0, \qquad |R| > a, \tag{4.2}$$

and $\Psi_1^* = o(R^2)$ as $|R| \to \infty$, together with,

$$\Psi_1^* = \tfrac{1}{2} U r^2, \qquad \frac{\partial \Psi_1^*}{\partial z} = 0 \quad \text{on} \quad |R| = a. \tag{4.3}$$

The correspondence principle (2.11) may now be applied to replace Ψ_1^* by Φ where

$$\Psi_1^* = \tfrac{1}{2} r^2 U \Phi \tag{4.4}$$

and Φ satisfies the equation

$$L_3^2 \Phi = 0, \tag{4.5}$$

together with the boundary conditions

$$\lim_{|R| \to \infty} \Phi = 0, \tag{4.6}$$

$$\Phi = 1, \qquad \frac{\partial \Phi}{\partial z} = 0 \quad \text{on} \quad |R| = a. \tag{4.7}$$

Once Φ has been determined, the original function Ψ_1 is given by

$$\Psi_1(r, z) = \tfrac{1}{2}Ur^2[1-\Phi(r, z)]. \tag{4.8}$$

The function Φ satisfying (4.5), (4.6) and (4.7) may be readily found by using a method quoted in Burns (1970). It has the form

$$\Phi(r, z) = \frac{3a}{2(r^2+z^2)^{\frac{1}{2}}}-\frac{a^3}{2(r^2+z^2)^{\frac{3}{2}}}.$$

Thus, employing (4.8)

$$\Phi_1(r, z) = \tfrac{1}{2}Ur^2\left[1+\frac{a^3}{2(r^2+z^2)^{\frac{3}{2}}}-\frac{3a}{2(r^2+z^2)^{\frac{1}{2}}}\right]$$

and the stress and displacement distributions are given by substitution in (3.4) and (3.5).

A physical interpretation for the above problem is readily given. Consider a block of incompressible elastic material with a rigid spherical adhering inclusion occupying the region $r^2+z^2 < a^2$ where a is small compared with the dimensions of the block. The block is resting on a rigid foundation on a plane z = constant and is subjected to a body force (such as gravity) acting in the z-direction and having negligible effect on the elastic material but a significant effect on the rigid inclusion. The body force causes the inclusion to be displaced by an amount U so that the displacement u_z is given by

$$\begin{aligned} u_z &= +U-\frac{1}{r}\frac{\partial\Psi_1}{\partial r} \\ &= \tfrac{3}{4}Uar^2[a^2(r^2+z^2)^{-\frac{5}{2}}-(r^2+z^2)^{-\frac{3}{2}}]+\tfrac{3}{2}Ua[a^2(r^2+z^2)^{-\frac{3}{2}}-(r^2+z^2)^{-\frac{1}{2}}], \end{aligned} \tag{4.9}$$

where a rigid body displacement $u_z = U$ has been superposed in order to satisfy the boundary conditions that $u_z \rightarrow 0$ at large distances from the origin and also that $u_z = U$ at the boundary of the spherical inclusion.

A similar analysis may be used for a rigid inclusion in the form of a spindle or torus.

5. Conclusion

It has been shown that an iterated form of a Baecklund transformation applied to the Stokes–Beltrami equations provides a means of solving certain inclusion problems in incompressible elasticity. Rigid inclusions in the shape of a sphere, spindle or torus may be treated in this manner. These solutions are of interest as a basis for the generation of second order solutions in second order incompressible finite elasticity (see Selvadurai & Spencer, 1972).

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