Engineering Analysis with Boundary Elements 29 (2005) 435–446

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

www.elsevier.com/locate/enganabound

A study of free terms for plate problems in the dual boundary integral equations

J.T. Chen[a,*], C.S. Wu[a], K.H. Chen[b]

[a]Department of Harbor and River Engineering, National Taiwan Ocean University, P.O. Box 7-59, Keelung 20224, Taiwan
[b]Department of Information Management, Toko University, Chia-Yi 613, Taiwan

Received 2 August 2004; revised 30 November 2004; accepted 20 December 2004
Available online 17 March 2005

Abstract

In this paper, we review the free terms of dual boundary integral equations for the Laplace and Navier equations of 2-D and 3-D problems and extend to biharmonic equation for plate problems. We derive the free terms of the dual BIE with a smooth boundary by means of the Taylor series expansion for the density through bump-contour technique surrounding the singularity. After using the limiting approach, the free terms and boundary terms for the 16 improper integrals in the dual formulation for the plate problems are derived. The contributions of single, double, triple and quadrapole potentials for the free term are also examined. The improper integrals due to the 16 kernels with singularity, hypersingularity or super-singularity are interpreted by the Cauchy principal value as well as finite parts.

Keywords: Dual boundary integral formulation; Free term; Boundary term; Bump-contour technique; Smooth boundary; Hypersingularity; Finite part

1. Introduction

Boundary integral equations (BIEs) with strongly singular and hypersingular kernels are currently employed in many fields of applied mechanics, most of the mathematical issues have been clarified for the evaluation of the singular integrals. The treatment of singularities has always been a key subject in the development of boundary element method (BEM). Dual boundary integral equations (DBIEs) for crack problems were derived using the limiting and trace approaches proposed by Hong and Chen [1]. Also, the DBIEs for the Laplace equation with a degenerate boundary was developed by Chen and Hong [2]. The numerical implementation has been termed the dual boundary element method by Portela et al. [22]. The dual formulation has been mainly applied to problems with a degenerate boundary by Chen and Hong in 1999, e.g. a screen in an acoustic cavity [5], a crack in an elastic body [16], and plate [24], thin airfoil in aerodynamics [23], combdrive in MEMs [18], a cutoff wall in potential flow [4], degenerate scales [25] and the adaptive BEM [7]. Mathematically speaking, the dual formuation can provide sufficient equations for a rank-deficiency system. Recently, the hypersingular equation has been utilized to provide a constraint at a corner in an analytical way by Gray and Elschner et al. Gray and Manne [11] have applied the hypersingular equation as an additional constraint to ensure a unique solution by a limiting process from an interior point to a corner. The three-dimensional case was also extended by Gray and Lutz [10]. How to determine the free terms in a hypersingular equation accurately has received attention in the dual BIE by Guiggiani [12–15]. Later, an additional free term in the hypersingular equation for the Laplace problem was independently obtained by Guiggiani [15] and Chen and Hong [3]. In 1995, Mantic and Paris [19] obtained the same results which corrected independently the error thus providing hypersingular boundary integral equations of potential problem. In 2000, Chen et al. [6,9] have proposed the bump-contour technique and the limiting approach to determine the free terms of the two- or three-dimensional Laplace and Navier equations successfully. Also, the free terms of dual BIE for the 2-D Helmholtz equation were presented [8]. Since the hypersingular

* Corresponding author. Fax: +886 2 2463 2932.
E-mail address: jtchen@ind.ntou.edu.tw (J.T. Chen).

0955-7997/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2004.12.006

Notations

$r \equiv \lvert s-x \rvert$ distance.	$\bar{t}_i, i=1,2$ normal vector of the field point.
$y_i - x_i - s_i$, $i=1,2$ vector component.	t_i, $i=1,2$ normal vector of the source point.
$\bar{n}_i$, $i=1,2$ normal vector of the field point.	ν poisson ratio.
n_i, $i=1,2$ normal vector of the source point.	

integral equation can provide an additional constraint for the Dirichlet problems, the free terms must be examined. Many researchers, for example, Guiggiani has derived the free terms in the boundary integral formulation by employing the direct method for the Laplace equation, the Navier equation and the biharmonic equation. He also found an additional free term for the corner problem using his approach instead of using the 'dual' formulation. Moreover, Maucher and Hartmann [20] studied the singularities of Kirchhoff plate for a boundary element solution. In 1994, Knöpke presented the derivation of a second-order gradient BIE [17], that is the identity for the bending moment components of elastic Kirchhoff plates.

In this paper, we focus on the fourth-order partial differentail equation, like bending of thin elastic plates where the BIE must face the improper integrals of hypersingular kernel or finte part. The order of super-singularity occurred in the dual formulation for plate problems is higher than that of hypersingularity. We derive the free terms on a smooth boundary by means of the bump-contour technique surrounding the singularity. After using the bump-contour technique and limiting approach, the free terms and boundary terms for the 16 improper integrals in the dual formulation are derived. The improper integrals due to the 16 kernels with weak singularity, strong singularity, hypersingularity and super-singularity are interpreted as the Cauchy principal value and finite parts.

2. Review of free terms of the dual integral formulation for 2-D and 3-D Laplace and Navier equations with a smooth boundary

According to the papers of Chen and his students as well as his colleagues [6,8,9], they derived the free terms of the dual integral equations in conjunction with the bump-contour technique and limiting process for the Laplace and Navier problems. First, let us consider the two- and three-dimensional Laplace equations with a smooth boundary point. The dual boundary integral equations are shown as follows:

2-D Laplace problem

$$2\pi u(x) = \int_{B'+B^-+B_\epsilon+B^+} [T(s,x)u(s) - U(s,x)t(s)]\mathrm{d}B(s), \quad x \in \Omega, \tag{1}$$

$$2\pi t(x) = \int_{B'+B^-+B_\epsilon+B^+} [M(s,x)u(s) - L(s,x)t(s)]\mathrm{d}B(s), \quad x \in \Omega, \tag{2}$$

3-D Laplace problem

$$4\pi u(x) = \int_{B'+B^-+B_\epsilon+B^+} [T(s,x)u(s) - U(s,x)t(s)]\mathrm{d}B(s), \quad x \in \Omega, \tag{3}$$

$$4\pi t(x) = \int_{B'+B^-+B_\epsilon+B^+} [M(s,x)u(s) - L(s,x)t(s)]\mathrm{d}B(s), \quad x \in \Omega, \tag{4}$$

where the U and M kernels are weakly singular and hypersingular kernel functions, respectively, while the T and L kernels are strongly singular kernel functions, B', B^-, B_ϵ and B^+ are the contour integration path including the singularity as shown in Fig. 1, and Ω is the domain of interest. In fact, the B integration path in Fig. 1 denotes the contour integration around the singularity with a radius ϵ, and $B^+ + B^- + B_\epsilon + B'$ is just the definition of the integration region of the Cauchy principal value. B^+ and B^- denote two of the elements in the B' boundary near the singularity as shown in Fig. 1. By adopting the bump-contour technique, we have the free terms and boundary terms and the results are summarized in Table 1. It is found that the contributions from both the hypersingular integrals and the strongly singular integrals for the free terms of the BIE are, respectively, half and half for the two-dimensional

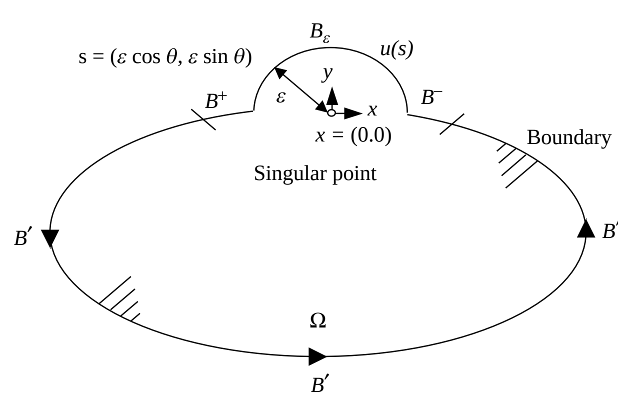

Fig. 1. The considered boundary integration path for the two-dimensional problem.

Table 1
Free terms of the dual BIEs for the 2-D and 3-D Laplace problems

2-D problem	
$U(s,x)$	$T(s,x)$
0	$\pi u(x)$
$L(s,x)$	$M(s,x)$
$-\frac{\pi}{2}t(x)$	$+\frac{\pi}{2}t(x)+\frac{2}{\epsilon}u(x)$
3-D problem	
$U(s,x)$	$T(s,x)$
0	$2\pi u(x)$
$L(s,x)$	$M(s,x)$
$-\frac{2\pi}{3}t(x)$	$\frac{4\pi}{3}t(x)-\frac{2\pi}{\epsilon}u(x)$

case, one-third and two-thirds for three-dimensional problem [6].

Secondly, the dual boundary integral equations for the 2-D and 3-D Navier equations are

$$u_i(x) = \int_{B'+B^-+B_\epsilon+B^+} [-U_{ki}(s,x)t_k(s) + T_{ki}(s,x)u_k(s)]\mathrm{d}B(s), \quad x\in\Omega, \tag{5}$$

$$t_i(x) = \int_{B'+B^-+B_\epsilon+B^+} [-L_{ki}(s,x)t_k(s) + M_{ki}(s,x)u_k(s)]\mathrm{d}B(s), \quad x\in\Omega, \tag{6}$$

where U_{ki}, T_{ki}, L_{ki} and M_{ki} are the four kernel functions which depend on the 2-D or 3-D case, the $u_k(s)$ and $t_k(s)$ are the kth components for the displacement and traction. After collecting the free terms and unbounded boundary terms, the dual boundary integral equations on a smooth boundary point for elasticity problems are derived without the problems of divergent integrals [9]. Similarly, the results were summarized in Table 2. It is found that single- and double-layer potentials contribute the free terms in the hypersingular equation. Comparing the results of the Laplace problem with those of the Navier equation, it is found that the free coefficients are the same, namely one half for the smooth boundary.

Table 2
Free terms of the dual BIEs for the 2-D and 3-D elasticity problems

2-D problem	$U_{ki}(s,x)$	$T_{ki}(s,x)$
$i=1,\ k=1$	No jump	$\frac{-u_1(x)}{2}$
$i=2,\ k=1$	No jump	0
$i=1,\ k=2$	No jump	0
$i=2,\ k=2$	No jump	$\frac{-u_2(x)}{2}$
	$L_{ki}(s,x)$	$M_{ki}(s,x)$
$i=1,\ k=1$	$\frac{G(3-4\nu)}{16(1-\nu)}\left\{\frac{\partial u_1}{\partial s_2}+\frac{\partial u_2}{\partial s_1}\right\}\Big\vert_{s=x}$	$\frac{-G}{8(1-\nu)}\,\frac{\partial u_1}{\partial s_2}\Big\vert_{s=x}$
$i=2,\ k=1$	$\frac{G(-1+4\nu)}{8(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_1}{\partial s_1}+\nu\frac{\partial u_2}{\partial s_2}\right\}\Big\vert_{s=x}$	$\frac{-G}{8(1-\nu)}\,\frac{\partial u_1}{\partial s_1}\Big\vert_{s=x}$
$i=1,\ k=2$	$\frac{G(3-4\nu)}{16(1-\nu)}\left\{\frac{\partial u_1}{\partial s_2}+\frac{\partial u_2}{\partial s_1}\right\}\Big\vert_{s=x}$	$\frac{-G}{8(1-\nu)}\,\frac{\partial u_2}{\partial s_1}\Big\vert_{s=x}$
$i=2,\ k=2$	$\frac{G(5-4\nu)}{8(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_2}{\partial s_2}+\nu\frac{\partial u_1}{\partial s_1}\right\}\Big\vert_{s=x}$	$\frac{-3G}{8(1-\nu)}\,\frac{\partial u_2}{\partial s_2}\Big\vert_{s=x}$
3-D problem		
	$U_{ki}(s,x)$	$T_{ki}(s,x)$
$i=1,\ k=1$	No jump	$\frac{-u_1(x)}{2}$
$i=2,\ k=1$	No jump	0
$i=3,\ k=1$	No jump	0
$i=1,\ k=2$	No jump	0
$i=2,\ k=2$	No jump	$\frac{-u_2(x)}{2}$
$i=3,\ k=2$	No jump	0
$i=1,\ k=3$	No jump	0
$i=2,\ k=3$	No jump	0
$i=3,\ k=3$	No jump	$\frac{-u_3(x)}{2}$
	$L_{ki}(s,x)$	$M_{ki}(s,x)$
$i=1,\ k=1$	$\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_1}{\partial s_3}+\frac{\partial u_3}{\partial s_1}\right\}\Big\vert_{s=x}$	$\frac{G(-7+5\nu)}{30(1-\nu)}\,\frac{\partial u_1}{\partial s_3}\Big\vert_{s=x}$
$i=2,\ k=1$	0	0
$i=3,\ k=1$	$\frac{G(-1+5\nu)}{15(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_1}{\partial s_1}+\nu\left(\frac{\partial u_2}{\partial s_2}+\frac{\partial u_3}{\partial s_3}\right)\right\}\Big\vert_{s=x}$	$-\frac{G(1+5\nu)}{15(1-\nu)}\,\frac{\partial u_1}{\partial s_1}\Big\vert_{s=x}$
$i=1,\ k=2$	0	0
$i=2,\ k=2$	$\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_2}{\partial s_3}+\frac{\partial u_3}{\partial s_2}\right\}\Big\vert_{s=x}$	$\frac{G(-7+5\nu)}{30(1-\nu)}\,\frac{\partial u_2}{\partial s_3}\Big\vert_{s=x}$
$i=3,\ k=2$	$\frac{G(-1+5\nu)}{15(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_2}{\partial s_2}+\nu\left(\frac{\partial u_1}{\partial s_1}+\frac{\partial u_3}{\partial s_3}\right)\right\}\Big\vert_{s=x}$	$-\frac{G(1+5\nu)}{15(1-\nu)}\,\frac{\partial u_2}{\partial s_2}\Big\vert_{s=x}$
$i=1,\ k=3$	$\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_1}{\partial s_3}+\frac{\partial u_3}{\partial s_1}\right\}\Big\vert_{s=x}$	$\frac{G(-7+5\nu)}{30(1-\nu)}\,\frac{\partial u_3}{\partial s_1}\Big\vert_{s=x}$
$i=2,\ k=3$	$\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_2}{\partial s_3}+\frac{\partial u_3}{\partial s_2}\right\}\Big\vert_{s=x}$	$\frac{G(-7+5\nu)}{30(1-\nu)}\,\frac{\partial u_3}{\partial s_2}\Big\vert_{s=x}$
$i=3,\ k=3$	$\frac{G(7-5\nu)}{15(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_3}{\partial s_3}+\nu\left(\frac{\partial u_1}{\partial s_1}+\frac{\partial u_2}{\partial s_2}\right)\right\}\Big\vert_{s=x}$	$\frac{-8G}{15(1-\nu)}\,\frac{\partial u_3}{\partial s_3}\Big\vert_{s=x}$

3. Free terms of the DBIEs with a smooth boundary for the biharmonic problems

The dual integral equations for the plate problem can be derived from the Rayleigh–Green identity as follows:

$$8\pi u(x) = \int_B [-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s)]\mathrm{d}B(s), \quad x\in\Omega, \tag{7}$$

$$8\pi\theta(x) = \int_B [-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)]\mathrm{d}B(s), \quad x\in\Omega, \tag{8}$$

$$8\pi m(x) = \int_B [-U_m(s,x)v(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s)]\mathrm{d}B(s), \quad x\in\Omega, \tag{9}$$

$$8\pi v(x) = \int_B [-U_v(s,x)v(s) + \Theta_v(s,x)m(s) - M_v(s,x)\theta(s) + V_v(s,x)u(s)]\mathrm{d}B(s), \quad x\in\Omega, \tag{10}$$

where B is the boundary, Ω is the domain of interest, u, θ, m and v mean the displacement, slope, normal moment and effective shear force, s and x are the source and field points, respectively. In Eqs. (1)–(6), we locate the point on the boundary, and introduce the integral equation for the domain point since B is modified to $B^+ + B^- + B_\epsilon + B'$ to embrace the boundary point x. In Eqs. (7)–(10), the collocation point is in the interior domain and B is the boundary, i.e. $B = \partial\Omega$. For the biharmonic equation, we can obtain the fundamental solution as follows

$$U(x,s) = U(s,x) = r^2 \ln(r), \tag{11}$$

where r is the distance between the field point and the source point written as $r \equiv |s-x|$. The other three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$, are defined as follows

$$\Theta(s,x) = \mathcal{K}_{\theta,s}(U(s,x)), \tag{12}$$

$$M(s,x) = \mathcal{K}_{m,s}(U(s,x)), \tag{13}$$

$$V(s,x) = \mathcal{K}_{v,s}(U(s,x)), \tag{14}$$

where $\mathcal{K}_{\theta,s}(\cdot)$, $\mathcal{K}_{m,s}(\cdot)$ and $\mathcal{K}_{v,s}(\cdot)$ mean the slope, moment and shear force operators with respect to s, respectively, which are defined as follows

$$\mathcal{K}_{\theta,s}(\cdot) = \frac{\partial(\cdot)}{\partial n_s}, \tag{15}$$

$$\mathcal{K}_{m,s}(\cdot) = \nu\nabla_s^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n_s^2}, \tag{16}$$

$$\mathcal{K}_{v,s}(\cdot) = \frac{\partial\nabla_s^2(\cdot)}{\partial n_s} + (1-\nu)\frac{\partial}{\partial t_s}\left[\left(\frac{\partial^2(\cdot)}{\partial n_s\,\partial t_s}\right)\right], \tag{17}$$

where ν is the Poisson's ratio, n and t are the normal and tangential vectors, respectively. By employing the bump-contour technique, the DBIEs in Eqs. (7)–(10) are derived as

$$8\pi u(x) = \int_{B'+B^-+B_\epsilon+B^+} [-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s)]\mathrm{d}B(s), \quad x \in B, \tag{18}$$

$$8\pi\theta(x) = \int_{B'+B^-+B_\epsilon+B^+} [-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)]\mathrm{d}B(s), \quad x \in B, \tag{19}$$

$$8\pi m(x) = \int_{B'+B^-+B_\epsilon+B^+} [-U_m(s,x)v(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s)]\mathrm{d}B(s), \quad x \in B, \tag{20}$$

$$8\pi v(x) = \int_{B'+B^-+B_\epsilon+B^+} [-U_v(s,x)v(s) + \Theta_v(s,x)m(s) - M_v(s,x)\theta(s) + V_v(s,x)u(s)]\mathrm{d}B(s), \quad x \in B, \tag{21}$$

where B', B^-, B_ϵ and B^+ are the contour integration paths including the domain Ω surrounding the singularity as shown in Fig. 1. For convenience, it was assumed that B_ϵ is an arc of a semi-circle centered at the field point x with radius ϵ. The integration path B_ϵ denotes the contour integration around the singular point, and $B'+B^-+B^+$ is the definition of the integration region of the Cauchy principal value. Eqs. (18)–(21) can be rewritten as

$$\begin{aligned} 8\pi u(x) &= \mathrm{CPV}\int_B [-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) \\ &\quad + V(s,x)u(s)]\mathrm{d}B(s) + \int_{B_\epsilon} [-U(s,x)v(s) + \Theta(s,x)m(s) \\ &\quad - M(s,x)\theta(s) + V(s,x)u(s)]\mathrm{d}B_\epsilon(s), \end{aligned} \tag{22}$$

$$\begin{aligned} 8\pi\theta(x) &= \mathrm{CPV}\int_B [-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) \\ &\quad + V_\theta(s,x)u(s)]\mathrm{d}B(s) + \int_{B_\epsilon} [-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) \\ &\quad - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)]\mathrm{d}B_\epsilon(s), \end{aligned} \tag{23}$$

$$\begin{aligned} 8\pi m(x) &= \mathrm{CPV}\int_B [-U_m(s,x)v(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) \\ &\quad + V_m(s,x)u(s)]\mathrm{d}B(s) + \int_{B_\epsilon} [-U_m(s,x)v(s) + \Theta_m(s,x)m(s) \\ &\quad - M_m(s,x)\theta(s) + V_m(s,x)u(s)]\mathrm{d}B_\epsilon(s), \end{aligned} \tag{24}$$

$$\begin{aligned} 8\pi v(x) &= \mathrm{CPV}\int_B [-U_v(s,x)v(s) + \Theta_v(s,x)m(s) - M_v(s,x)\theta(s) \\ &\quad + V_v(s,x)u(s)]\mathrm{d}B(s) + \int_{B_\epsilon} [-U_v(s,x)v(s) + \Theta_v(s,x)m(s) \\ &\quad - M_v(s,x)\theta(s) + V_v(s,x)u(s)]\mathrm{d}B_\epsilon(s), \end{aligned} \tag{25}$$

where the CPV is the Cauchy principal value. The rigorous definition of CPV will be elaborated on later. The integral over B_ϵ contributes to the free term. In each integrand, both kernel function and density function on B_ϵ will be studied in Sections 4 and 5.

4. Taylor expansion of boundary density functions

Before deriving the free terms of the improper integral equations, the density functions (displacement, slope, moment and shear force) are needed to be expanded to series form for order analysis. Therefore, we expand the density functions by using the Taylor series in the BIE formulation as follows:

The displacement, $u(s)$, is

$$u(s)=u(x)+\left[\frac{\partial u(s)}{\partial s_1}\cos\theta+\frac{\partial u(s)}{\partial s_2}\sin\theta\right]\epsilon+\frac{1}{2!}\left[\frac{\partial^2 u(s)}{\partial s_1^2}\cos^2\theta+\frac{\partial^2 u(s)}{\partial s_2^2}\sin^2\theta+\frac{\partial^2 u(s)}{\partial s_1\,\partial s_2}\cos\theta\sin\theta\right]\epsilon^2+\frac{1}{3!}\left[\frac{\partial^3 u(s)}{\partial s_1^3}\cos^3\theta+\frac{\partial^3 u(s)}{\partial s_1\,\partial s_2^2}\cos\theta\sin^2\theta+\frac{\partial^3 u(s)}{\partial s_2\,\partial s_1^2}\sin\theta\cos^2\theta+\frac{\partial^3 u(s)}{\partial s_2^3}\sin^3\theta\right]\epsilon^3+O(\epsilon^4)+\cdots \tag{26}$$

The slope, moment and effective shear force can be obtained by employing the operators in Eqs. (12)–(14), respectively. We have the slope

$$\theta(s)=\left[\frac{\partial u(s)}{\partial s_1}\cos\theta+\frac{\partial u(s)}{\partial s_2}\sin\theta\right]+\left[\frac{\partial^2 u(s)}{\partial s_1^2}\cos^2\theta+\frac{\partial^2 u(s)}{\partial s_2^2}\sin^2\theta+\frac{\partial^2 u(s)}{\partial s_1\,\partial s_2}\cos\theta\sin\theta\right]\epsilon+\frac{1}{2}\left[\frac{\partial^3 u(s)}{\partial s_1^3}\cos^3\theta+\frac{\partial^3 u(s)}{\partial s_1\,\partial s_2^2}\cos\theta\sin^2\theta+\frac{\partial^3 u(s)}{\partial s_2\,\partial s_1^2}\sin\theta\cos^2\theta+\frac{\partial^3 u(s)}{\partial s_2^3}\sin^3\theta\right]\epsilon^2+O(\epsilon^3)+\cdots \tag{27}$$

The bending moment is

$$m(s)=\frac{\partial^2 u(s)}{\partial s_1^2}[\cos^2\theta+\nu\sin^2\theta]+\frac{\partial^2 u(s)}{\partial s_2^2}[\sin^2\theta+\nu\cos^2\theta]+\left\{\frac{\partial^3 u(s)}{\partial s_1^3}[\cos^3\theta+\nu\cos\theta\sin^2\theta]+\frac{\partial^3 u(s)}{\partial s_1\,\partial s_2^2}\left[\frac{\nu}{3}\cos^3\theta+\left(1-\frac{2\nu}{3}\right)\cos\theta\sin^2\theta\right]\times\frac{\partial^3 u(s)}{\partial s_2\,\partial s_1^2}\left[\frac{\nu}{3}\sin^3\theta+\left(1-\frac{2\nu}{3}\right)\sin\theta\cos^2\theta\right]+\frac{\partial^3 u(s)}{\partial s_2^3}[\cos^3\theta+\nu\cos\theta\sin^2\theta]\right\}\epsilon+O(\epsilon^2)+\cdots \tag{28}$$

The effective shear force is

$$v(s)=\frac{1-\nu}{\epsilon}\left[-\frac{\partial^2 u(s)}{\partial s_1^2}(\cos^2\theta-\sin^2\theta)\frac{\partial^2 u(s)}{\partial s_2^2}\times(\cos^2\theta-\sin^2\theta)-2\frac{\partial^2 u(s)}{\partial s_1\,\partial s_2}\cos\theta\sin\theta\right]+\frac{\partial^3 u(s)}{\partial s_1^3}[\cos\theta+(1-\nu)(\cos^3\theta-2\cos\theta\sin^2\theta)]+\frac{1}{3}\frac{\partial^3 u(s)}{\partial s_1\,\partial s_2^2}[\cos\theta+(1-\nu)(2\cos^3\theta-7\cos\theta\sin^2\theta)]+\frac{1}{3}\frac{\partial^3 u(s)}{\partial s_2\,\partial s_1^2}[\sin\theta+(1-\nu)(2\sin^3\theta-7\cos^2\theta\sin\theta)]+\frac{\partial^3 u(s)}{\partial s_2^3}[\sin\theta+(1-\nu)(2\sin\theta\cos^2\theta-\sin^3\theta)]+O(\epsilon)+\cdots \tag{29}$$

The density functions are the Taylor expansions at x and they should be substituted into the dual integral equations when deriving the free terms. The simplified forms of the density functions, $u(x)$, $\theta(x)$, $m(x)$ and $v(x)$, under the condition of $n_x=(0,1)$ and $t_x=(-1,0)$ are shown in Table 3 without loss of generality.

5. Explicit forms for the kernel functions and the order analysis for the asymptotic behavior

Sixteen kernel functions of the boundary integral equations are very lengthy and are summarized in Appendix A. By adopting the boundary integral formulations and the 16 kernel functions, the notations generally employed in the Kirchhoff plate theory are briefly summarized. Without loss of generality, we have the following notations as shown in Fig. 1:

(1) The position of the field point: $x=(x_1,x_2)=(0,0)$.
(2) The position of the source point: $s=(s_1,s_2)=(\epsilon\cos\theta,\epsilon\sin\theta)$.
(3) Distance: $r=|s-x|$.
(4) Vector component: $y_i=x_i-s_i$, $i=1,2$.
(5) Normal vector of the field point: $n(x)=(\bar{n}_1,\bar{n}_2)=(0,1)$.
(6) Normal vector of the source point along the arc: $n(s)=(n_1,n_2)=(\cos\theta,\sin\theta)$.

Table 3
Simplified forms of the density functions

Displacement	$u(x)$
Slope	$\theta(x)=\frac{\partial u(x)}{\partial n_x}=\frac{\partial u(x)}{\partial x_2}$
Moment	$m(x)=\nu\nabla^2 u(x)+(1-\nu)\frac{\partial^2 u(x)}{\partial n_x^2}=\nu\frac{\partial^2 u(x)}{\partial x_1^2}+\frac{\partial^2 u(x)}{\partial x_2^2}$
Shear force	$v(x)=\frac{\partial\nabla^2 u(x)}{\partial n_x}+(1-\nu)\frac{\partial}{\partial t_x}\left[\frac{\partial^2 u(x)}{\partial n_x\,\partial t_x}\right]=(2-\nu)\frac{\partial^3 u(x)}{\partial x_1^2\,\partial x_2}+\frac{\partial^2 u(x)}{\partial x_2^3}$

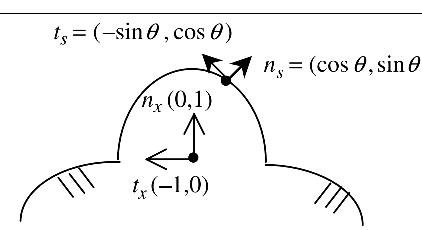

Table 4
Order analysis for the 16 kernels of biharmonic problem

$U(s,x)$	$\Theta(s,x)$	$M(s,x)$	$V(s,x)$
$O(\epsilon^2 \ln \epsilon)$	$O(\epsilon \ln \epsilon)$	$O(\ln \epsilon)$	$O(\frac{1}{\epsilon})$
$U_\theta(s,x)$	$\Theta_\theta(s,x)$	$M_\theta(s,x)$	$V_\theta(s,x)$
$O(\epsilon \ln \epsilon)$	$O(\ln \epsilon)$	$O(\frac{1}{\epsilon})$	$O\left(\frac{1}{\epsilon^2}\right)$
$U_m(s,x)$	$\Theta_\theta(s,x)$	$M_m(s,x)$	$V_m(s,x)$
$O(\ln \epsilon)$	$O(\frac{1}{\epsilon})$	$O\left(\frac{1}{\epsilon^2}\right)$	$O\left(\frac{1}{\epsilon^3}\right)$
$U_v(s,x)$	$\Theta_v(s,x)$	$M_v(s,x)$	$V_v(s,x)$
$O(\frac{1}{\epsilon})$	$O\left(\frac{1}{\epsilon^2}\right)$	$O\left(\frac{1}{\epsilon^3}\right)$	$O\left(\frac{1}{\epsilon^4}\right)$

(7) Tangential vector of the field point: $t(x)=(\bar{t}_1,\bar{t}_2)=(-1,0)$.
(8) Tangential vector of the source point: $t(s)=(t_1,t_2)=(-\sin\theta,\cos\theta)$.

By employing the bump-contour technique and substituting the notations of (1)–(8) in Fig. 1, we can derive the explicit forms of the 16 kernels of the dual integral formulation. From the asymptotic analysis of the above-mentioned kernels, the order analysis for the asymptotic behavior in the kernels can be found in Table 4.

6. Potential due to the 16 kernels for the bump integral

After defining the related symbols, 16 kernel funtions and the density functions for the bump-contour technique, we substitute them into the boundary integral formulations in Eqs. (7)–(10) and derive the free terms and boundary terms. The four improper integrals of the first dual integral formulation are

$$(u1):\int_{B_\epsilon} U(s,x)v(s)\mathrm{d}B(s)=\epsilon^3\ln(\epsilon)\quad\text{(finite value)},\tag{30}$$

$$(u2):\int_{B_\epsilon} \Theta(s,x)m(s)\mathrm{d}B(s)=\epsilon^2\ln(\epsilon)\quad\text{(finite value)},\tag{31}$$

$$(u3):\int_{B_\epsilon} M(s,x)\theta(s)\mathrm{d}B(s)=\epsilon\ln\epsilon\quad\text{(finite value)},\tag{32}$$

$$(u4):\int_{B_\epsilon} V(s,x)u(s)\mathrm{d}B(s)=4\pi u(x)+\epsilon\quad\text{(finite value)},\tag{33}$$

as ϵ approaches zero, the free term is $4\pi u(x)$. It is worth noting that the free term only occurs in the V kernel, the other kernels do not contribute any values to the free terms.

For the second equation of the dual formulation, we have

$$(\theta 1):\int_{B_\epsilon} U_\theta(s,x)v(s)\mathrm{d}B(s)=\epsilon^3\ln\epsilon\quad\text{(finite value)},\tag{34}$$

$$(\theta 2):\int_{B_\epsilon} \Theta_\theta(s,x)m(s)\mathrm{d}B(s)=\epsilon\ln\epsilon\quad\text{(finite value)},\tag{35}$$

$$(\theta 3):\int_{B_\epsilon} M_\theta(s,x)\theta(s)\mathrm{d}B(s)$$

$$=-\pi(1+\nu)\theta(x)+\epsilon\quad\text{(finite value)}.\tag{36}$$

By integrating the $M_\theta(s,x)$ kernel, the free term is found to depend on the Poisson ratio. We may wonder the existence of the Poisson ratio in terms of the successful experiences of the Laplace equation, the reasonable explanation is attributed to the operator of kernel functions.

$$(\theta 4):\int_{B_\epsilon} V_\theta(s,x)u(s)\mathrm{d}B(s)=(3-\nu)\pi\theta(x)+\frac{4(3-\nu)}{\epsilon}u(x),\tag{37}$$

where the former one is the free term and the latter one is the unbounded boundary term. Apparently, the free terms of the second integral equation are different from those of the first one that the contribution of the free term is composed of the M_θ and V_θ and the boundary term only results from the V_θ integral in Eq. (37). By collecting the two free terms in Eqs. (36) and (37), a sum of $4\pi\theta(x)$ is obtained. Then, let us consider the higher-order gradient boundary integral equation of the third boundary integral formulation as follows:

$$(m1):\int_{B_\epsilon} U_m(s,x)v(s)\mathrm{d}B(s)=\frac{\pi}{2}(\nu-1)m(x)+O(\epsilon),\tag{38}$$

$$(m2):\int_{B_\epsilon} \Theta_m(s,x)m(s)\mathrm{d}B(s)=\pi(1+\nu)m(x)+O(\epsilon),\tag{39}$$

$$(m3):\int_{B_\epsilon} M_m(s,x)\theta(s)\mathrm{d}B(s)$$

$$=-\pi m(x)-4(1-\nu)\left(1+\frac{5\nu}{3}\right)\frac{\theta(x)}{\epsilon}+O(\epsilon),\tag{40}$$

$$(m4):\int_{B_\epsilon} V_m(s,x)u(s)\mathrm{d}B(s)$$

$$=\frac{\pi}{2}(3-\nu)m(x)+\frac{8(1-\nu)(3-\nu)}{3}\frac{\theta(x)}{\epsilon}+O(\epsilon).\tag{41}$$

By taking the limit, the $O(\epsilon)$ term approaches zero. After obtaining the free terms, we find that the free terms are contributed by all the four kernels in the third boundary integral equation. It is interesting that the sum of free terms is also found to be $4\pi m(x)$ in companion with some unbounded terms. In view of the order analysis of the four integral equations, we may wonder why the free term of U_m occurs? To the authors' best knowledge, we may explain that the result is attributed to the complex form of the density function $v(s)$.

$$(v1): \int_{B_\epsilon} U_v(s,x)v(s)\mathrm{d}B(s) = \frac{\pi(\nu+1)(\nu-3)}{2}v(x) + O(\epsilon), \tag{42}$$

$$(v2): \int_{B_\epsilon} \Theta_v(s,x)m(s)\mathrm{d}B(s) = 2\pi v(x) + \left[\frac{4(-\nu^2+6\nu+7)}{3}\right]\frac{m(x)}{\epsilon} + O(\epsilon), \tag{43}$$

$$(v3): \int_{B_\epsilon} M_v(s,x)\theta(s)\mathrm{d}B(s) = -\pi(\nu-1)(\nu-2)v(x) + \left[\frac{8(\nu-1)(\nu+7)}{3\epsilon}\right]m(x) + \left[\frac{(1-\nu)}{3\epsilon^2}[3\pi(\nu-5)+16\nu]\right]\theta(x) + O(\epsilon), \tag{44}$$

$$(v4): \int_{B_\epsilon} V_v(s,x)u(s)\mathrm{d}B(s) = -\frac{\pi}{2}(\nu-1)(\nu-3)v(x) + \left[\frac{4(\nu-1)(\nu-1)}{\epsilon}\right]m(x) + \left[\frac{1-\nu}{3\epsilon^2}[3\pi(\nu-5)+16(\nu-3)]\right]\theta(x) + \left[\frac{8(\nu-1)^2}{\epsilon^3}\right]u(x) + O(\epsilon). \tag{45}$$

According to the integrals in Eqs. (42)–(45), we find that the free terms are contributed by all the kernels and the sum is $4\pi v(x)$. It is interesting to find that the order descends in a successive order for the 16 integrals. After combining the CPV and the boundary terms to obtain the finite value as coined by the finite part, Eqs. (18)–(21) are rewritten as

$$4\pi u(x) = -\mathrm{FP}\int_B U(s,x)v(s)\mathrm{d}B(s) + \mathrm{FP}\int_B \Theta(s,x)m(s)\mathrm{d}B(s) - \mathrm{FP}\int_B M(s,x)\theta(s)\mathrm{d}B(s) + \mathrm{FP}\int_B V(s,x)u(s)\mathrm{d}B(s), \tag{46}$$

$$4\pi\theta(x) = -\mathrm{FP}\int_B U_\theta(s,x)v(s)\mathrm{d}B(s) + \mathrm{FP}\int_B \Theta_\theta(s,x)m(s)\mathrm{d}B(s) - \mathrm{FP}\int_B M_\theta(s,x)\theta(s)\mathrm{d}B(s) + \mathrm{FP}\int_B V_\theta(s,x)u(s)\mathrm{d}B(s), \tag{47}$$

$$4\pi m(x) = -\mathrm{FP}\int_B U_m(s,x)v(s)\mathrm{d}B(s) + \mathrm{FP}\int_B \Theta_m(s,x)m(s)\mathrm{d}B(s) - \mathrm{FP}\int_B M_m(s,x)\theta(s)\mathrm{d}B(s) + \mathrm{FP}\int_B V_m(s,x)u(s)\mathrm{d}B(s), \tag{48}$$

$$4\pi v(x) = -\mathrm{FP}\int_B U_v(s,x)v(s)\mathrm{d}B(s) + \mathrm{FP}\int_B \Theta_v(s,x)m(s)\mathrm{d}B(s) - \mathrm{FP}\int_B M_v(s,x)\theta(s)\mathrm{d}B(s) + \mathrm{FP}\int_B V_v(s,x)u(s)\mathrm{d}B(s), \tag{49}$$

in which

$$\mathrm{FP}\int_B U(s,x)v(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} U(s,x)v(s)\mathrm{d}B(s), \tag{50}$$

$$\mathrm{FP}\int_B \Theta(s,x)m(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} \Theta(s,x)m(s)\mathrm{d}B(s), \tag{51}$$

$$\mathrm{FP}\int_B M(s,x)\theta(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} M(s,x)\theta(s)\mathrm{d}B(s), \tag{52}$$

$$\mathrm{FP}\int_B V(s,x)u(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} V(s,x)u(s)\mathrm{d}B(s), \tag{53}$$

$$\mathrm{FP}\int_B U_\theta(s,x)v(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} U_\theta(s,x)v(s)\mathrm{d}B(s), \tag{54}$$

$$\mathrm{FP}\int_B \Theta_\theta(s,x)m(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} \Theta_\theta(s,x)m(s)\mathrm{d}B(s), \tag{55}$$

$$\mathrm{FP}\int_B M_\theta(s,x)\theta(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} M_\theta(s,x)\theta(s)\mathrm{d}B(s), \tag{56}$$

Table 5
Free terms due to the bump integral for the biharmonic equation

$U(s,x)$	$\Theta(s,x)$	$M(s,x)$	$V(s,x)$
0	0	0	$[4\pi]u(x)$
$U_\theta(s,x)$	$\Theta_\theta(s,x)$	$M_\theta(s,x)$	$V_\theta(s,x)$
0	0	$[-\pi(1+\nu)]\theta(x)$	$[(3-\nu)\pi]\theta(x)+[4(3-\nu)]\frac{u(x)}{\epsilon}$
$U_m(s,x)$	$\Theta_m(s,x)$	$M_m(s,x)$	$V_m(s,x)$
$\left[\frac{\pi}{2}(\nu-1)\right]m(x)$	$[\pi(1+\nu)]m(x)$	$[-\pi]m(x)+\left[-4(1-\nu)+(1+\frac{5}{3}\nu)\right]\frac{\theta(x)}{\epsilon}$	$\left[\frac{\pi}{2}(3-\nu)\right]m(x)+\left[\frac{8(1-\nu)(3-\nu)}{3}\right]\frac{\theta(x)}{\epsilon}+\frac{0}{\epsilon^2}u(x)$
$U_v(s,x)$	$\Theta_v(s,x)$	$M_v(s,x)$	$V_v(s,x)$
$\left[\frac{\pi}{2}(\nu+1)(\nu-3)\right]$ $v(x)$	$[2\pi]v(x)+$ $\left[\frac{4}{3}(-\nu^2+6\nu+7)\right]$ $\frac{m(x)}{\epsilon}$	$[-\pi(-1+\nu)(-2+\nu)]v(x)+\left[\frac{8(\nu-1)(\nu+7)}{3}\right]\frac{m(x)}{\epsilon}+$ $\left[\frac{(1-\nu)}{3}[3\pi(\nu-5)+16\nu]\right]\frac{\theta(x)}{\epsilon^2}$	$\left[\frac{-\pi}{2}(-1+\nu)(-3+\nu)\right]v(x)+[4(\nu-1)(\nu-1)]\times$ $\frac{m(x)}{\epsilon}+\left[\frac{(1-\nu)}{3}[3\pi(\nu-5)+16(\nu-3)]\right]\frac{\theta(x)}{\epsilon^2}+$ $[8(-1+\nu)^2]\frac{u(x)}{\epsilon^3}$

$$\mathrm{FP}\int_B V_\theta(s,x)u(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} V_\theta(s,x)u(s)\mathrm{d}B(s)+\frac{4(3-\nu)}{\epsilon}u(x), \quad (57)$$

$$\mathrm{FP}\int_B U_m(s,x)v(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} U_m(s,x)v(s)\mathrm{d}B(s), \quad (58)$$

$$\mathrm{FP}\int_B \Theta_m(s,x)m(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} \Theta_m(s,x)m(s)\mathrm{d}B(s),x \quad (59)$$

$$\mathrm{FP}\int_B M_m(s,x)\theta(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} M_m(s,x)\theta(s)\mathrm{d}B(s) -4(1-\nu)\left(1+\frac{5\nu}{3}\right)\frac{\theta(x)}{\epsilon}, \quad (60)$$

$$\mathrm{FP}\int_B V_m(s,x)u(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} V_m(s,x)u(s)\mathrm{d}B(s)+\left[\frac{8(\nu-1)(\nu+7)}{3\epsilon}\right]m(x) +\left[\frac{(1-\nu)}{3\epsilon^2}[3\pi(\nu-5)+16\nu]\right]\theta(x), \quad (61)$$

$$\mathrm{FP}\int_B U_v(s,x)v(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} U_v(s,x)v(s)\mathrm{d}B(s), \quad (62)$$

$$\mathrm{FP}\int_B \Theta_v(s,x)m(s)\mathrm{d}B(s) = \mathrm{CPV}\int_{B'+B^-+B^+} \Theta_v(s,x)m(s)\mathrm{d}B(s) +\left[\frac{4(-\nu^2+6\nu+7)}{3}\right]\frac{m(x)}{\epsilon}, \quad (63)$$

$$\mathrm{FP}\int_B M_v(s,x)\theta(s)\mathrm{d}B(s) = \mathrm{CPV}\int_B M_v(s,x)\theta(s)\mathrm{d}B(s)+\left[\frac{8(\nu-1)(\nu+7)}{3\epsilon}\right]m(x) +\left[\frac{(1-\nu)}{3\epsilon^2}[3\pi(\nu-5)+16\nu]\right]\theta(x), \quad (64)$$

$$\mathrm{FP}\int_B V_v(s,x)u(s)\mathrm{d}B(s) = \mathrm{CPV}\int_B V_v(s,x)u(s)\mathrm{d}B(s)+\left[\frac{4(\nu-1)(\nu-1)}{\epsilon}\right]m(x) +\left[\frac{(1-\nu)}{3\epsilon^2}[3\pi(\nu-5)+16(\nu-3)]\right]\theta(x) +\left[\frac{8(\nu-1)^2}{\epsilon^3}\right]u(x), \quad (65)$$

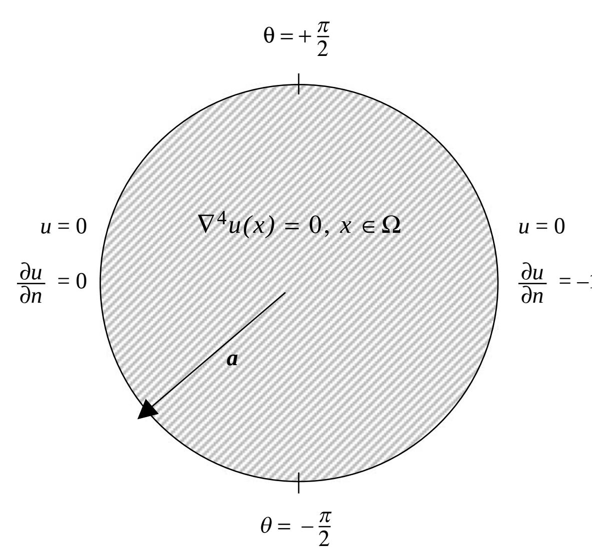

Fig. 2. The chart of the biharmonic equation with the essential boundary condition.

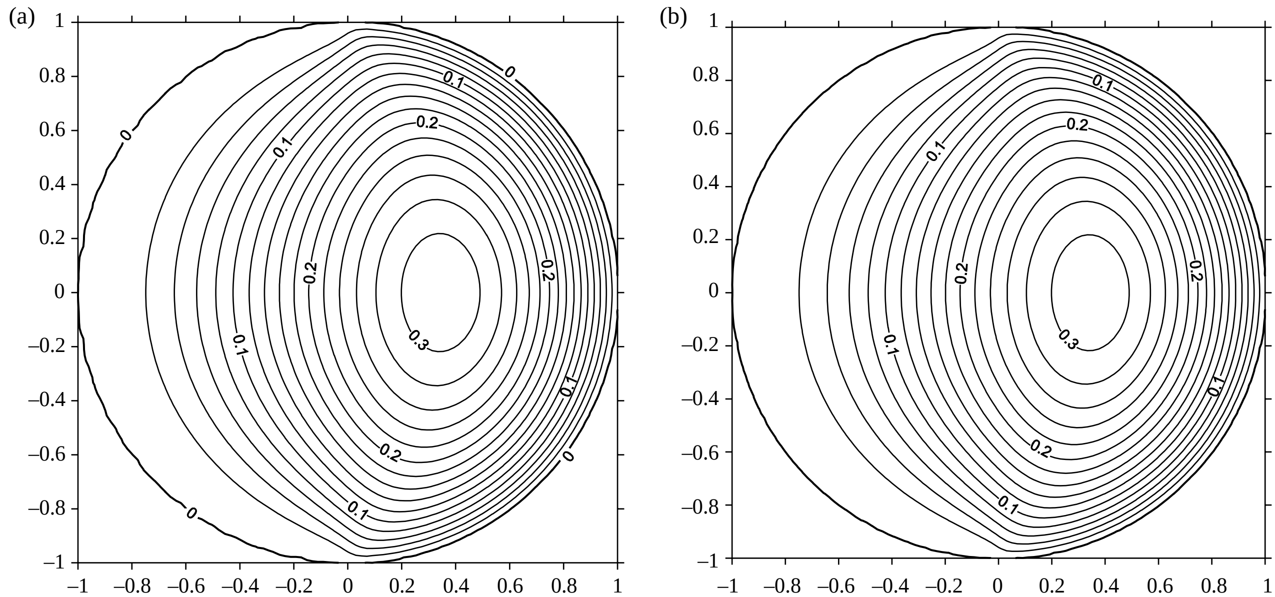

Fig. 3. The contour plots of the biharmonic fields with the essential boundary condition using the boundary element method. (a) Number of boundary elements, 42; (b) number of boundary elements, 102.

where CPV and FP denote the Cauchy principal value and finite part, respectively. For the biharmonic problem, we have the kernels with higher singularity than the hypersingularity which is termed unnamed singularity by Guiggiani. Here, the unnamed singularity is called super-singularity. Therefore, we denote them as the 'finite part' in a unified manner. The boundary terms of the kernel integration arise from the boundary integral equations naturally and can compensate the infinity of CPV. Combining the 16 improper integrals, we have the boundary integral equations with the free coefficient of 4π for a smooth boundary. The free terms are contributed from different kernels instead of only one Cauchy kernel. Finally, all the results of free terms and boundary terms are summarized in Table 5.

To check the validity of the present formulation, the BEM was utilized to solve the following problem [21] in Fig. 2. The governing equation is

$$\nabla^4 u(x) = 0, \quad x \in \Omega, \tag{66}$$

subject to the essential boundary conditions

$$u(r,\theta)|_{r=a} = 0, \quad 0<\theta<2\pi$$

$$\left.\frac{\partial u(r,\theta)}{\partial r}\right|_{r=a} = \begin{cases} -1, & -\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2}<\theta<\dfrac{3\pi}{2} \end{cases} \tag{67}$$

where Ω is a circular domain with radius $a=1$. The exact solution was avaliable by Mills [21] as follows

$$u(r,\theta) = \frac{1}{2\pi}(1-r^2)\left[\gamma + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\frac{\pi}{2}-\theta}{2}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\frac{\pi}{2}-\theta}{2}\right)\right)\right] \tag{68}$$

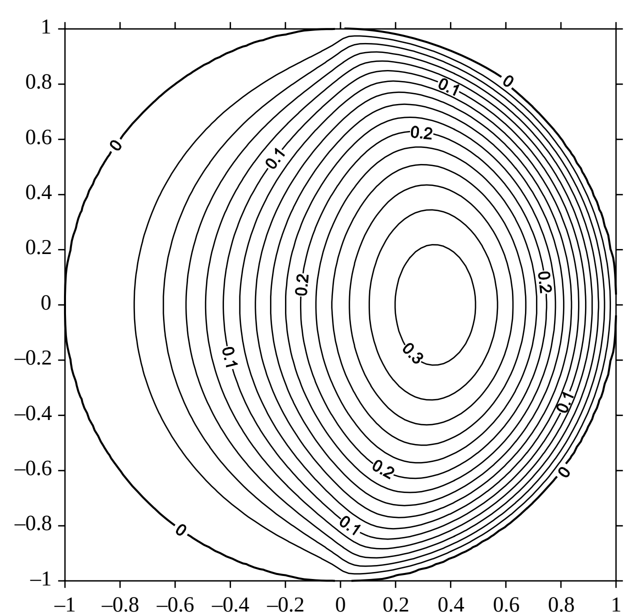

Fig. 4. The contour plot of the exact solution.

where

$$\gamma = \begin{cases} 0, & \dfrac{-\pi}{2} < \theta < \dfrac{\pi}{2} \\ \pi, & \dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2} \end{cases}. \tag{69}$$

Also, the series solution is available [25]

$$u(r,\theta) = \frac{1}{4}(1-r^2) - \sum_{m=1}^{\infty} \frac{1}{m\theta} \sin\left(\frac{m\theta}{2}\right)(r^m \cos(m\theta) - r^{m+2}\cos(m\theta)). \tag{70}$$

The numerical results were plotted by using 42 and 102 constant elememts as shown in Fig. 3(a) and (b). After comparing with the exact solution in Fig. 4, the BEM results agree well with the exact solution.

7. Conclusions

In this paper, the free terms of the DBIEs for the biharmonic problem were derived successfully. We adopted the bump-contour technique surrounding the singularity and expanded the density functions by using the Taylor series. Contribution of the single, double, triple and quadrapole potentials on the free terms was determined. After combinig the boundary term with the Cauchy principal value, finite part was obtained in the derivation. After collecting the 16 improper integrals for the smooth boundary, it is interesting to find that the sum of the free terms in each boundary integral equations is 4π. Finally, the order analysis and the free terms of the 16 kernels of the biharmonic equation are summarized in Table 5. The potentials of the 16 kernels can be interpreted as finite part and Cauchy principal value. In addition, a numerical example was tested to see the validity of the formulation.

Acknowledgements

Financial support from the National Science Council under Grant No.NSC-93-2211-E-019-010 for National Taiwan Ocean University is gratefully acknowledged.

Appendix A. Sixteen kernel functions for the biharmonic problem

Kernels for the first equation of the boundary integral formulations:
$8\pi u(x) = \int_B \{-U(x,s)v(s) + \Theta(x,s)m(s) - M(s,x)\theta(s) + V(s,x)u(s)\} \mathrm{d}B(s)$

$$U(s,x) = r^2 \ln r$$

$$\Theta(s,x) = -(1 + 2\ln r)y_i n_i$$

$$M(s,x) = \nu\left[\frac{2}{r^2} y_i y_i + 2(1 + 2\ln r)\right] + (1-\nu)\left[\frac{2}{r^2} y_i n_i y_j n_j + (1 + 2\ln r) n_i n_i\right]$$

$$V(s,x) = \left[\frac{4}{r^4} y_i y_i y_j n_j - \frac{8}{r^2} y_j n_j\right] + (1-\nu)\left[\frac{4}{r^4} y_j n_j y_i t_i y_k t_k + \frac{2}{r^3} y_i t_i y_j t_j - \frac{2}{r^2} y_j n_j t_i t_i - \frac{2}{r^3} y_j n_j y_i n_i\right]$$

Kernels for the second equation of the boundary integral formulations:
$8\pi\theta(x) = \int_B \{-U_\theta(x,s)v(s) + \Theta_\theta(x,s)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\} \mathrm{d}B(s)$

$$U_\theta(s,x) = (1 + 2\ln r) y_i \bar{n}_i$$

$$\Theta_\theta(s,x) = -\left[\frac{2}{r^2} y_i \bar{n}_i y_j n_j + (1 + 2\ln r) n_i \bar{n}_i\right]$$

$$M_\theta(s,x) = 4\nu\left[\frac{-1}{r^4} y_i y_i y_k \bar{n}_k + \frac{2}{r^2} y_i \bar{n}_i\right] + 2(1-\nu)\left[\frac{-2}{r^4} y_i n_i y_j n_j y_k \bar{n}_k + \frac{3}{r^2} y_i n_i n_j \bar{n}_j + \frac{2}{r^2} y_k \bar{n}_k n_i n_i\right]$$

$$\begin{aligned} V_\theta(s,x) = {} & \left[\frac{-16}{r^6} y_i y_i y_j n_j y_l \bar{n}_l + \frac{24}{r^4} y_i n_i y_j \bar{n}_j + \frac{4}{r^4} y_i y_i n_j \bar{n}_j - \frac{8}{r^2} n_j \bar{n}_j\right] + (1-\nu)\left[\frac{-16}{r^6} y_i t_i y_j n_j y_k t_k y_l \bar{n}_l + \frac{4}{r^4} y_i t_i y_k t_k n_j \bar{n}_j + \frac{8}{r^4} y_j n_j y_k t_k \bar{n}_i t_i - \frac{4}{r^5} y_i t_i y_j t_j y_l \bar{n}_l \right. \\ & \left. + \frac{4}{r^3} y_j t_j \bar{n}_i t_i + \frac{4}{r^4} y_j n_j t_i t_i y_l \bar{n}_l - \frac{2}{r^2} t_i t_i n_j \bar{n}_j + \frac{4}{r^5} y_i n_i y_j n_j y_l \bar{n}_l - \frac{4}{r^3} y_i n_i n_j \bar{n}_j\right] \end{aligned}$$

Kernels for the third equation of the boundary integral formulations:
$8\pi m(x) = \int_B \{-U_m(x,s)v(s) + \Theta_m(x,s)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s)\}\mathrm{d}B(s)$

$$U_m(s,x) = \nu\left[\frac{2}{r^2}y_iy_i + 2(1+2\ln r)\right] + (1-\nu)\left[\frac{2}{r^2}y_i\bar{n}_iy_jn_j + (1+2\ln r)\bar{n}_i\bar{n}_i\right]$$

$$\Theta_m(s,x) = 4\nu\left[\frac{1}{r^4}y_iy_iy_kn_k - \frac{2}{r^2}y_in_i\right] + 2(1-\nu)\left[\frac{-2}{r^4}y_i\bar{n}_iy_j\bar{n}_jy_kn_k - \frac{3}{r^2}y_i\bar{n}_in_j\bar{n}_j - \frac{2}{r^2}y_kn_k\bar{n}_i\bar{n}_i\right]$$

$$\begin{aligned} M_m(s,x) = \nu\Bigg\{&16\nu\left[\frac{1}{r^6}y_iy_iy_ky_k - \frac{2}{r^4}y_iy_i + \frac{1}{r^2}\right] + (1-\nu)\left[\frac{16}{r^6}y_in_iy_jn_jy_ky_k - \frac{24}{r^4}y_in_iy_jn_j - \frac{4}{r^4}y_ky_kn_in_i + \frac{8}{r^2}n_in_i\right]\Bigg\} \\ &+ (1-\nu)\Bigg\{16\nu\left[\frac{4}{r^6}y_iy_iy_k\bar{n}_ky_l\bar{n}_l + \left(\frac{-2}{r^4}\right)y_k\bar{n}_ky_i\bar{n}_i + \left(\frac{-1}{r^4}\right)y_iy_i\bar{n}_k\bar{n}_k + \left(\frac{-4}{r^3}\right)y_i\bar{n}_iy_l\bar{n}_l + \left(\frac{2}{r^2}\right)\bar{n}_i\bar{n}_i\right] \\ &+2(1-\nu)\left[\frac{8}{r^5}y_in_iy_jn_jy_k\bar{n}_ky_l\bar{n}_l + \left(\frac{-4}{r^4}\right)y_jn_jy_k\bar{n}_kn_i\bar{n}_i + \left(\frac{-2}{r^4}\right)y_in_iy_jn_j\bar{n}_k\bar{n}_k + \left(\frac{-6}{r^3}\right)y_in_iy_l\bar{n}_ln_j\bar{n}_j + \left(\frac{-4}{r^3}\right)y_k\bar{n}_ky_l\bar{n}_ln_in_i \right. \\ &\left. + \left(\frac{3}{r^2}\right)n_j\bar{n}_jn_i\bar{n}_i + \left(\frac{2}{r^2}\right)n_in_i\bar{n}_k\bar{n}_k\right]\Bigg\} \end{aligned}$$

$$\begin{aligned} V_m(s,x) = \nu&\left[\frac{96}{r^8}y_iy_iy_jn_jy_ly_l - \frac{192}{r^6}y_iy_iy_jn_j + \frac{96}{r^4}y_jn_j\right] + \nu(1-\nu)\left[\frac{96}{r^8}y_it_iy_jn_jy_kt_ky_ly_l - \frac{128}{r^6}y_jn_jy_kt_ky_it_i + \frac{16}{r^7}y_i\bar{t}_iy_j\bar{t}_j + \frac{24}{r^4}y_jn_jt_kt_k - \frac{24}{r^5}y_it_iy_jt_j + \frac{4}{r^3}t_jt_j \right. \\ &\left. - \frac{16}{r^6}y_jn_jt_it_iy_ly_l - \frac{16}{r^7}y_jn_jy_in_iy_ly_l + \frac{24}{r^5}y_in_iy_jn_j - \frac{4}{r^3}n_in_i\right] + (1-\nu)\left[\frac{96}{r^8}y_iy_iy_jn_jy_l\bar{n}_ly_p\bar{n}_p - \frac{128}{r^6}y_in_iy_j\bar{n}_jy_l\bar{n}_l - \frac{32}{r^6}y_iy_iy_l\bar{n}_ln_j\bar{n}_j - \frac{16}{r^6}y_iy_iy_jn_j\bar{n}_l\bar{n}_l \right. \\ &\left. + \frac{24}{r^4}y_jn_j\bar{n}_i\bar{n}_i + \frac{48}{r^4}y_i\bar{n}_in_j\bar{n}_j\right] + (1-\nu)(1-\nu)\left[\frac{96}{r^8}y_it_iy_jn_jy_kt_ky_l\bar{n}_ly_p\bar{n}_p - \frac{32}{r^6}y_jn_jy_kt_ky_l\bar{n}_l\bar{n}_it_i - \frac{16}{r^6}y_it_iy_kt_ky_l\bar{n}_ln_j\bar{n}_j - \frac{16}{r^6}y_it_iy_jn_jy_kt_k\bar{n}_l\bar{n}_l \right. \\ & - \frac{32}{r^6}y_jn_jy_kt_k\bar{n}_it_iy_p\bar{n}_p + \frac{16}{r^4}y_jn_j\bar{n}_it_in_k\bar{n}_k + \frac{8}{r^4}y_jn_j\bar{n}_it_i\bar{n}_kt_k - \frac{16}{r^6}y_it_iy_kt_kn_j\bar{n}_jy_p\bar{n}_p + \frac{16}{r^7}y_it_iy_jt_jy_l\bar{n}_ly_p\bar{n}_p - \frac{16}{r^5}y_jt_jy_l\bar{n}_l\bar{n}_it_i - \frac{4}{r^5}y_it_iy_jt_j\bar{n}_l\bar{n}_l + \frac{4}{r^3}\bar{n}_it_i\bar{n}_jt_j \\ & - \frac{16}{r^6}y_jn_jy_l\bar{n}_lt_it_iy_p\bar{n}_p + \frac{16}{r^7}y_it_iy_jt_jy_l\bar{n}_ly_p\bar{n}_p - \frac{16}{r^5}y_jt_jy_l\bar{n}_l\bar{n}_it_i - \frac{4}{r^5}y_it_iy_jt_j\bar{n}_l\bar{n}_l + \frac{4}{r^3}\bar{n}_it_i\bar{n}_jt_j - \frac{16}{r^6}y_jn_jy_l\bar{n}_lt_it_iy_p\bar{n}_p + \frac{4}{r^4}y_jn_jt_it_i\bar{n}_l\bar{n}_l + \frac{4}{r^4}y_l\bar{n}_lt_it_in_j\bar{n}_j \\ &\left. - \frac{16}{r^7}y_jn_jy_in_iy_l\bar{n}_ly_p\bar{n}_p + \frac{8}{r^5}y_in_iy_l\bar{n}_ln_j\bar{n}_j + \frac{4}{r^5}y_jn_jy_in_i\bar{n}_l\bar{n}_l + \frac{4}{r^4}n_j\bar{n}_jt_it_iy_p\bar{n}_p + \frac{8}{r^5}y_in_in_j\bar{n}_jy_p\bar{n}_p - \frac{4}{r^3}n_i\bar{n}_in_j\bar{n}_j\right] \end{aligned}$$

Kernels for the fourth equation of the boundary integral formulations:
$8\pi v(x) = \int_B \{-U_v(x,s)v(s) + \Theta_v(x,s)m(s) - M_v(s,x)\theta(s) + V_v(s,x)u(s)\}\mathrm{d}B(s)$

$$U_v(s,x) = \left[\frac{-4}{r^4}y_iy_iy_j\bar{n}_j + \frac{8}{r^2}y_j\bar{n}_j\right] + (1-\nu)\left[\frac{-4}{r^4}y_j\bar{n}_jy_i\bar{t}_iy_k\bar{t}_k + \frac{2}{r^3}y_i\bar{t}_iy_j\bar{t}_j + \frac{2}{r^2}y_j\bar{n}_j\bar{t}_i\bar{t}_i - \frac{2}{r^3}y_j\bar{n}_jy_i\bar{n}_i\right]$$

$$\begin{aligned} \Theta_v(s,x) = &\left[\frac{-16}{r^6}y_iy_iy_j\bar{n}_jy_ln_l + \frac{24}{r^4}y_in_iy_j\bar{n}_j + \frac{4}{r^4}y_iy_in_j\bar{n}_j - \frac{8}{r^2}n_j\bar{n}_j\right] + (1-\nu)\left[\frac{-16}{r^6}y_i\bar{t}_iy_j\bar{n}_jy_k\bar{t}_ky_ln_l + \frac{4}{r^4}y_i\bar{t}_iy_k\bar{t}_kn_j\bar{n}_j + \frac{8}{r^4}y_j\bar{n}_jy_k\bar{t}_kn_i\bar{t}_i + \frac{4}{r^5}y_i\bar{t}_iy_j\bar{t}_jy_ln_l \right. \\ &\left. - \frac{4}{r^3}y_j\bar{t}_jn_i\bar{t}_i + \frac{4}{r^4}y_j\bar{n}_j\bar{t}_i\bar{t}_iy_ln_l - \frac{2}{r^2}\bar{t}_i\bar{t}_in_j\bar{n}_j - \frac{4}{r^5}y_i\bar{n}_iy_j\bar{n}_jy_ln_l + \frac{4}{r^3}y_i\bar{n}_in_j\bar{n}_j\right] \end{aligned}$$

$$\begin{aligned} M_v(s,x) = \nu&\left[\frac{-96}{r^8}y_iy_iy_j\bar{n}_jy_ly_l + \frac{192}{r^6}y_iy_iy_j\bar{n}_j - \frac{96}{r^4}y_j\bar{n}_j\right] + \nu(1-\nu)\left[\frac{-96}{r^8}y_i\bar{t}_iy_j\bar{n}_jy_k\bar{t}_ky_ly_l + \frac{128}{r^6}y_j\bar{n}_jy_k\bar{t}_ky_i\bar{t}_i + \frac{16}{r^7}y_i\bar{t}_iy_j\bar{t}_j - \frac{24}{r^4}y_j\bar{n}_j\bar{t}_k\bar{t}_k - \frac{24}{r^5}y_i\bar{t}_iy_j\bar{t}_j + \frac{4}{r^3}\bar{t}_j\bar{t}_j \right. \\ &\left. + \frac{16}{r^6}y_j\bar{n}_j\bar{t}_i\bar{t}_iy_ly_l - \frac{16}{r^7}y_j\bar{n}_jy_i\bar{n}_iy_ly_l + \frac{24}{r^5}y_i\bar{n}_iy_j\bar{n}_j - \frac{4}{r^3}\bar{n}_i\bar{n}_i\right] + (1-\nu)\left[\frac{-96}{r^8}y_iy_iy_j\bar{n}_jy_ln_ly_pn_p + \frac{128}{r^6}y_in_iy_j\bar{n}_jy_ln_l + \frac{32}{r^6}y_iy_iy_ln_ln_j\bar{n}_j + \frac{16}{r^6}y_iy_iy_j\bar{n}_jn_ln_l \right. \\ &\left. - \frac{24}{r^4}y_j\bar{n}_jn_in_i - \frac{48}{r^4}y_in_in_j\bar{n}_j\right] + (1-\nu)(1-\nu)\left[\frac{-96}{r^8}y_i\bar{t}_iy_j\bar{n}_jy_k\bar{t}_ky_ln_ly_pn_p + \frac{32}{r^6}y_j\bar{n}_jy_k\bar{t}_ky_ln_ln_i\bar{t}_i - \frac{16}{r^6}y_i\bar{t}_iy_k\bar{t}_ky_ln_ln_j\bar{n}_j + \frac{16}{r^6}y_i\bar{t}_iy_j\bar{n}_jy_k\bar{t}_kn_ln_l \right. \\ & + \frac{32}{r^6}y_j\bar{n}_jy_k\bar{t}_kn_i\bar{t}_iy_pn_p - \frac{16}{r^4}y_j\bar{n}_jn_i\bar{t}_in_kn_k - \frac{8}{r^4}y_j\bar{n}_jn_i\bar{t}_in_k\bar{t}_k + \frac{16}{r^6}y_i\bar{t}_iy_k\bar{t}_kn_j\bar{n}_jy_pn_p + \frac{16}{r^7}y_i\bar{t}_iy_j\bar{t}_jy_ln_ly_pn_p - \frac{16}{r^5}y_j\bar{t}_jy_ln_ln_i\bar{t}_i - \frac{4}{r^5}y_i\bar{t}_iy_j\bar{t}_jn_ln_l + \frac{4}{r^3}n_i\bar{t}_in_j\bar{t}_j \\ & + \frac{16}{r^6}y_j\bar{n}_jy_ln_l\bar{t}_i\bar{t}_iy_pn_p + \frac{16}{r^7}y_i\bar{t}_iy_j\bar{t}_jy_ln_ly_pn_p - \frac{16}{r^5}y_j\bar{t}_jy_ln_ln_i\bar{t}_i - \frac{4}{r^5}y_i\bar{t}_iy_j\bar{t}_jn_ln_l + \frac{4}{r^3}n_i\bar{t}_in_j\bar{t}_j + \frac{16}{r^6}y_j\bar{n}_jy_ln_l\bar{t}_i\bar{t}_iy_pn_p - \frac{4}{r^4}y_j\bar{n}_j\bar{t}_i\bar{t}_in_ln_l - \frac{4}{r^4}y_ln_l\bar{t}_i\bar{t}_in_j\bar{n}_j \\ &\left. - \frac{16}{r^7}y_j\bar{n}_jy_i\bar{n}_iy_ln_ly_pn_p + \frac{8}{r^5}y_i\bar{n}_iy_ln_ln_j\bar{n}_j + \frac{4}{r^5}y_j\bar{n}_jy_i\bar{n}_in_ln_l - \frac{4}{r^4}n_j\bar{n}_j\bar{t}_i\bar{t}_iy_pn_p + \frac{8}{r^5}y_i\bar{n}_in_j\bar{n}_jy_pn_p - \frac{4}{r^3}n_i\bar{n}_in_j\bar{n}_j\right] \end{aligned}$$

$$
\begin{aligned}
V_\nu(s,x) = {} & \Bigg[\frac{-768}{r^{10}} y_i y_i y_j \bar{n}_j y_l y_l y_p n_p + \frac{1536}{r^8} y_i n_i y_j \bar{n}_j y_l y_l + \frac{96}{r^8} y_i y_i y_l y_l n_j \bar{n}_j - \frac{192}{r^6} y_i y_i n_j \bar{n}_j - \frac{768}{r^6} y_i n_i y_j \bar{n}_j + \frac{96}{r^4} n_j \bar{n}_j \Bigg] + (1-\nu)\Bigg[\frac{-768}{r^{10}} y_i \bar{t}_i y_j \bar{n}_j y_k \bar{t}_k y_l y_l y_p n_p \\
& + \frac{192}{r^8} n_i \bar{t}_i y_j \bar{n}_j y_k \bar{t}_k y_l y_l + \frac{96}{r^8} y_i \bar{t}_i n_j \bar{n}_j y_k \bar{t}_k y_l y_l + \frac{960}{r^8} y_j \bar{n}_j y_k \bar{t}_k y_p n_p + \frac{96}{r^9} y_i \bar{t}_i y_j \bar{t}_j y_l y_l y_p n_p + \frac{96}{r^8} y_j \bar{n}_j \bar{t}_i \bar{t}_i y_l y_l y_p n_p - \frac{96}{r^8} y_i \bar{t}_i y_j \bar{n}_j y_k \bar{t}_k y_p n_p - \frac{128}{r^6} n_j \bar{n}_j y_i \bar{t}_i y_k \bar{t}_k \\
& - \frac{256}{r^6} y_i \bar{t}_i y_j \bar{n}_j n_k \bar{t}_k - \frac{32}{r^7} n_i \bar{t}_i y_j \bar{t}_j y_l y_l - \frac{32}{r^7} y_i \bar{t}_i y_j \bar{t}_j y_l n_l - \frac{128}{r^6} y_j \bar{n}_j \bar{t}_k \bar{t}_k y_p n_p - \frac{96}{r^7} y_i \bar{t}_i y_j \bar{t}_j y_p n_p - \frac{16}{r^6} n_j \bar{n}_j \bar{t}_i \bar{t}_i y_l y_l + \frac{32}{r^7} n_j \bar{n}_j y_i \bar{n}_i y_l y_l + \frac{128}{r^7} y_j \bar{n}_j y_i \bar{n}_i y_l n_l \\
& + \frac{24}{r^4} n_j \bar{n}_j \bar{t}_k \bar{t}_k + \frac{48}{r^5} n_j \bar{t}_j y_i \bar{t}_i + \frac{8}{r^5} y_p n_p \bar{t}_k \bar{t}_k - \frac{48}{r^5} n_i \bar{n}_i y_j \bar{n}_j - \frac{8}{r^5} \bar{n}_i \bar{n}_i y_p n_p \Bigg] + (1-\nu)\Bigg[\frac{96}{r^8} y_i y_i y_j \bar{n}_j t_l t_l y_p n_p + \frac{96}{r^8} y_i y_i y_j \bar{n}_j y_l n_l y_p n_p - \frac{128}{r^6} y_i n_i y_j \bar{n}_j t_l t_l \\
& - \frac{128}{r^7} y_i n_i y_j \bar{n}_j y_l n_l - \frac{16}{r^6} y_i y_i n_j \bar{n}_j t_l t_l - \frac{32}{r^7} y_i y_i n_j \bar{n}_j y_l n_l + \frac{24}{r^4} t_i t_i n_j \bar{n}_j + \frac{48}{r^5} y_i n_i n_j \bar{n}_j \Bigg] + (1-\nu)(1-\nu) + (1-\nu)\Bigg[+(1-\nu)\Bigg[\frac{96}{r^8} y_i y_i y_j \bar{n}_j t_l t_l y_p n_p \\
& + \frac{96}{r^8} y_i y_i y_j \bar{n}_j y_l n_l y_p n_p - \frac{128}{r^6} y_i n_i y_j \bar{n}_j t_l t_l - \frac{128}{r^7} y_i n_i y_j \bar{n}_j y_l n_l - \frac{16}{r^6} y_i y_i n_j \bar{n}_j t_l t_l - \frac{32}{r^7} y_i y_i n_j \bar{n}_j y_l n_l + \frac{24}{r^4} t_i t_i n_j \bar{n}_j + \frac{48}{r^5} y_i n_i n_j \bar{n}_j \\
& + (1-\nu)(1-\nu)\Bigg[-\frac{32}{r^6} t_i \bar{n}_i y_j n_j t_k \bar{t}_k y_l \bar{t}_l - \frac{32}{r^6} t_i \bar{t}_i y_j n_j t_k \bar{t}_k y_p \bar{n}_p + \frac{8}{r^4} t_i \bar{t}_i n_j \bar{n}_j t_k \bar{t}_k - \frac{32}{r^6} t_i \bar{t}_i y_j n_j t_k \bar{n}_k y_p \bar{n}_p + \frac{8}{r^4} t_i \bar{t}_i n_j \bar{t}_j t_k \bar{n}_k - \frac{8}{r^5} t_i \bar{n}_i y_j n_j t_k \bar{n}_k + \frac{8}{r^5} t_i \bar{t}_i y_j n_j t_k \bar{t}_k \\
& + \frac{8}{r^4} t_i \bar{n}_i n_j \bar{t}_j t_k \bar{t}_k - \frac{8}{r^5} t_j \bar{n}_j t_i \bar{t}_i y_l \bar{t}_l - \frac{8}{r^5} t_i \bar{t}_i t_j \bar{t}_j y_p \bar{n}_p - \frac{8}{r^5} t_i \bar{t}_i t_j \bar{n}_j y_q \bar{t}_q - \frac{4}{r^4} t_i \bar{n}_i t_j \bar{n}_j + \frac{4}{r^4} t_i \bar{t}_i t_j \bar{t}_j - \frac{16}{r^6} t_i t_i n_j \bar{t}_j y_p \bar{n}_p y_q \bar{t}_q - \frac{4}{r^5} t_i t_i n_j \bar{n}_j y_p \bar{n}_p + \frac{4}{r^5} t_i t_i n_j \bar{t}_j y_p \bar{t}_p \\
& + \frac{96}{r^9} y_i n_i y_j n_j y_l \bar{t}_l y_p \bar{n}_p y_q \bar{t}_q - \frac{32}{r^7} n_i \bar{t}_i y_j n_j y_l \bar{t}_l y_p \bar{n}_p - \frac{16}{r^7} y_i n_i y_j n_j \bar{t}_l \bar{t}_l y_p \bar{n}_p + \frac{16}{r^8} y_i n_i y_j n_j y_l \bar{n}_l y_p \bar{n}_p - \frac{16}{r^8} y_i n_i y_j n_j y_l \bar{t}_l y_p \bar{t}_p - \frac{32}{r^7} n_i \bar{n}_i y_j n_j y_l \bar{t}_l y_q \bar{t}_q \\
& + \frac{8}{r^5} n_i \bar{n}_i n_j \bar{t}_j y_l \bar{t}_l + \frac{8}{r^5} n_i \bar{n}_i y_j n_j \bar{t}_l \bar{t}_l + \frac{8}{r^6} n_i \bar{t}_i y_j n_j y_l \bar{t}_l - \frac{8}{r^6} n_i \bar{n}_i y_j n_j y_l \bar{n}_l - \frac{32}{r^7} n_i \bar{t}_i y_j n_j y_p \bar{n}_p y_q \bar{t}_q + \frac{8}{r^5} n_i \bar{t}_i n_j \bar{t}_j y_p \bar{n}_p - \frac{8}{r^6} n_i \bar{n}_i y_j n_j y_p \bar{n}_p + \frac{8}{r^6} n_i \bar{t}_i y_j n_j y_p \bar{t}_p \\
& + \frac{8}{r^5} n_i \bar{t}_i n_j \bar{n}_j y_q \bar{t}_q + \frac{4}{r^4} n_i \bar{n}_i n_j \bar{n}_j - \frac{4}{r^4} n_i \bar{t}_i \bar{n}_j \bar{t}_j + \frac{96}{r^8} t_i t_i y_j n_j y_l \bar{t}_l y_p \bar{n}_p y_q \bar{t}_q - \frac{16}{r^6} t_i t_i n_j \bar{t}_j y_l \bar{t}_l y_p \bar{n}_p - \frac{16}{r^6} t_i t_i y_j n_j \bar{t}_l \bar{t}_l y_p \bar{n}_p \\
& + \frac{16}{r^7} t_i t_i y_j n_j y_l \bar{n}_l y_p \bar{n}_p - \frac{16}{r^7} t_i t_i y_j n_j y_l \bar{t}_l y_p \bar{t}_p - \frac{16}{r^6} t_i t_i n_j \bar{n}_j y_l \bar{t}_l y_q \bar{t}_q + \frac{4}{r^4} t_i t_i n_j \bar{n}_j \bar{t}_l \bar{t}_l + \frac{4}{r^5} t_i t_i n_j \bar{t}_j y_l \bar{t}_l - \frac{4}{r^5} t_i t_i n_j \bar{n}_j y_l \bar{n}_l \Bigg]
\end{aligned}
$$

References

[1] Chen JT. On Hadamard principal value and boundary integral formulation of fracture mechanics, Master Thesis, Institute of Applied Mechanics, National Taiwan University; 1986.

[2] Chen JT, Hong H-K. Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. Appl Mech Rev 1999;52(1):17–33.

[3] Chen JT, Hong H-K. Dual boundary integral equations at a corner using contour approach around singularity. Adv Eng Soft 1994;21: 169–78.

[4] Chen JT, Hong H-K, Chyuan SW. Boundary element analysis and design in seepage flow problems with sheetpiles. Finite Element Anal Des 1994;17:1–20.

[5] Chen JT, Chen KH. Dual integral formualtion for determining the acoustic modes of a two-dimensional cavity with a degenerate boundary. Eng Anal Bound Element 1998;21(2):105–16.

[6] Chen JT, Kuo SR, Chen WC, Liu LW. On the free terms of the dual BEM for two and three-dimensional Laplace problems. J Mar Sci Tech 2000;8(1):8–15.

[7] Chen JT, Chen KH, Chen CT. Adaptive boundary element method of time-harmonic exterior acoustics in two-dimensions. Comput Methods Appl Mech Eng 2002;191:3331–45.

[8] Chen IL, Liang MT, Kuo SR, Chen JT. Dual boundary integral equation for Helmholtz equation at a corner using contour approach around singularity. J Mar Sci Tech 2001;9(1):53–63.

[9] Chen JT, Chen WC, Chen KH, Chen IL. Revisit of the free terms of the dual BIEs for elasticity. Kuwait J Sci Tech 2003; 30(2):1–22.

[10] Gray LJ, Lutz E. On the treatment of corners in the boundary element method. J Comput Appl Math 1990;32:369–86.

[11] Gray LJ, Manne LL. Hypersingular integrals at a corner. Eng Anal Bound Element 1993;11:327–34.

[12] Guiggiani M, Casalini P. Direct computation of Cauchy principal value integrals in advanced boundary elements. Int J Numer Methods Eng 1987;24:1711–6.

[13] Guiggiani M, Gigante A. A general algorithm for multidimensional Cauchy principal value integrals in the boundary element method. J Appl Mech, ASME 1990;57:906–15.

[14] Guiggiani M. The evaluation of Cauchy principal value integrals in the boundary element method—a review. Math Comp Model Spl Issue BIEM and BEM 1991;15:175–84.

[15] Guiggiani M. Hypersingular boundary integral equations have an additional free term. Comput Mech 1995;16:245–8.

[16] Hong H-K, Chen JT. Derivation of integral equations in elasticity. J Eng Mech, ASCE 1988;114(6):1028–44.

[17] Knöpke B. The hypersingular integral equation for the bending moment m_{xx} m_{xy} m_{yy} of the Kirchhoff plate. Comp Mech 1994;15:19–30.

[18] Liao YS, Chyuan SW, Chen JT. An alternatively efficient method for simulating the electrostatic field and levitating force of MEMS combdrive. J Micromech Microeng 2004;14:1258–69.

[19] Mantic V, Paris F. Existence and evaluation of the two free terms in the hypersingular boundary integral equation of potential theory. Eng Anal Bound Element 1995;16:253–60.

[20] Maucher R, Hartmann F. Corner singularities of Kirchhoff plate and boundary element method. Comput Methods Appl Mech Eng 1999; 173:273–85.

[21] Mills RD. Computing internal viscous flow problems for the circle by integral methods. J Fluid Mech 1977;12:609–24.

[22] Portela A, Aliabadi MH, Rooke DP. The dual boundary element method: effective implementation for crack problems. Int J Num Methods Eng 1992;33:1269–87.

[23] Wang CS, Chu S, Chen JT. Boundary element method for predicting store airloads during its carriage and separation procedure. In: Grilli S, Brebbia CA, Cheng AHD, editors. Computational Engineering Bound Elements, vol. 1, 1990. pp. 305–17

[24] Wearing JL, Ahmadi-Brooghani SY. The evaluation of stress intensity factors in plate bending problems using the dual boundary element method. Eng Anal Bound Element 1999;23(1):3–19.

[25] Wu CS. Degenerate scale analysis for membrane and plate problems using the meshless method and boundary element method, Master Thesis, Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung; 2004.