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A NEW FORMULA FOR PRINCIPAL AXES

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Introduction

To compute principal axes of a two-dimensional state of stress, the current technique is to either a double angle-tangent formula or draw a Mohr circle which requires a new sign convention for shear stress other than that used in the classical mechanics theory. The two angles from the double angle-tangent formula do not provide the principal axes directly, one has to verify individually the angles that correspond to the respective principal stresses. A new formula is presented here that is not found anywhere in print by the author which will provide uniquely the principal axis corresponding to a particular principal stress. This formula is very convenient for determining the principal axes in computer codes. The formula can be extended for analysis of principal strains for plane strain condition also.

Classical Plane Stress Problem and Mohr Circle

It is well known that the transformation of a two dimensional plane stress at point is studied by using a wedge of an element inclined at angle θ to the x-axis and to determine the principal stresses, the equations for stress components are transformed into angles of 2θ [1,2].

$$\sigma_{x'} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

Either maximizing $\sigma_{x'}$ or setting $\tau_{x'y'}$ equal to zero, we have the directions of the principal axes as

$$\tan 2\theta = \frac{2\tau_{xy}}{\left(\sigma_x - \sigma_y\right)} \qquad (3)$$

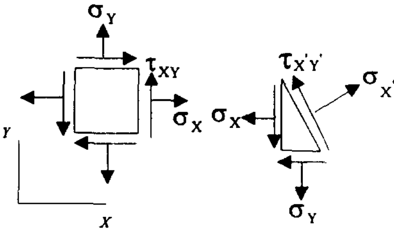

Fig. 1. Transformation of Stresses

which has two values θ_1 and θ_2 that are the directions of the principal stresses. However, in order to determine the angle that corresponds to the particular principal stress, one has to substitute the value of the angle in Eq. (1) and verify whether it is matching. An alternate approach is to draw the Mohr circle. In order to do that Eqs.(1) and (2) are rewritten in a different form in terms of σ- and τ- coordinates and eliminating the angle 2θ, we have the equation for Mohr circle, as

$$(\sigma - c)^2 + \tau^2 = A^2 + B^2 = r^2, \tag{4}$$

with center, $c = \frac{1}{2}(\sigma_x + \sigma_y)$ and radius $r = \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}$. And the principal stresses are:

$$\sigma_{1,2} = c \pm r = \frac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{5}$$

Since we have transformed the stress equations into functions of 2θ, any rotation θ of the coordinate axes represents a rotation equal to 2θ on the Mohr circle. This transformation forces one to use a new shear stress convention to plot stresses on the Mohr circle, a sign convention which is contradictory to the classical shear stress convention in mechanics. However, from the Mohr circle it is easy to determine the directions of the principal axes and the concept of Mohr circle is very good, if properly understood. Even though these sign conventions are only for plotting the stresses on the Mohr circle, they are confusing to many, because they are presented differently in text books by different authors [1,2]. Besides, these conventions are to be ignored after their use in drawing the Mohr circle.

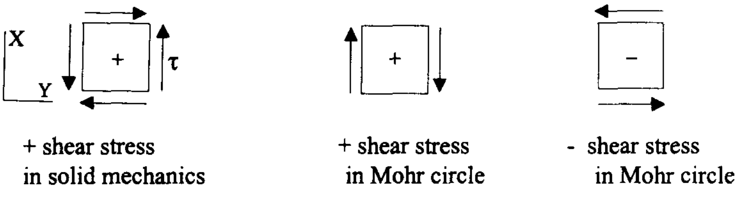

Fig. 2. Sign Conventions for Shear Stress

A New Formula for Principal Axes

New formula is developed [3] for the directions of the σ_1 and σ_2 and they are given by the following equations

$$\tan\theta_1 = \frac{\tau_{xy}}{\sigma_1 - \sigma_y} = \frac{\sigma_1 - \sigma_x}{\tau_{xy}} \quad \text{and} \quad \tan\theta_2 = \frac{\tau_{xy}}{\sigma_2 - \sigma_y} = \frac{\sigma_2 - \sigma_x}{\tau_{xy}} \tag{7}$$

where θ_1 and θ_2 are angles between the x- axis and the principal stress directions. Also, it may be noted that $\theta_1 = \theta_2 \pm 90$. These equations uniquely gives the principal directions and are very convenient for writing computer codes. The advantages of this formula are:

1. simple and unique equations for principal axes,
2. no new sign conventions for shear stresses,
3. directly useful in computer codes, and
4. applicable for plane strain condition also.

Derivation of the New Formula

Since principal stresses are eigen values of the stress tensor, we can write the eigen value problem as

$$\begin{bmatrix} \sigma_x - \sigma_1 & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma_1 \end{bmatrix} \begin{Bmatrix} c \\ s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{9}$$

where $c = \cos \theta_1$ and $s = \sin \theta_1$ Solving for θ_1, we have the equations shown in Eq. (7) above.

A Numerical Example.

The following two dimensional plane stress at a point is given:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} -7 & -3 \\ -3 & 1 \end{bmatrix} \text{MPa.}$$

Solution:

The principal stresses are:

$$\sigma_{1,2} = \frac{1}{2}(-7+1) \pm \sqrt{\frac{1}{4}(-7-1)^2 + (-3)^2}$$

$$= +2.0 \text{ and } -8 \text{ MPa.}$$

The principal axis for σ_1 is given by the Eq.(2.9)

$$\tan\theta_1 = \frac{\tau_{xy}}{\sigma_1 - \sigma_y} = \frac{-3}{2-1} = -3$$

and $\theta_1 = -71.57^0$. Similarly, the principal axis

for σ_2 is given by $\tan\theta_2 = \dfrac{\tau_{xy}}{\sigma_2 - \sigma_y} = \dfrac{-3}{-8-1} = .0.3333$

and $\theta_2 = +18.43^0$. The directions are illustrated in the Fig. 3.

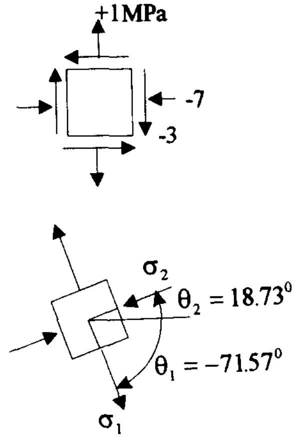

Fig. 3. Directions of Principal Stresses

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