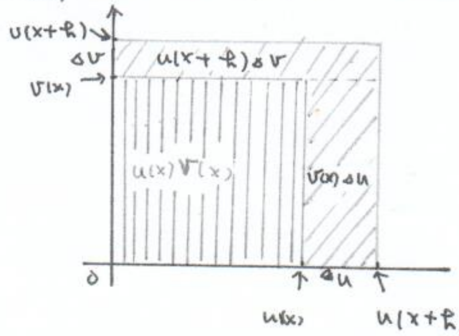
Derivative product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Geometric interpretation for derivative product rule



Suppose $u(x)$ and $v(x)$ are positive and increase when x increases, and $h>0$. Then the change in the product uv is the difference in areas of the larger and smaller "squares," which is the sum of the upper and right-hand reddish-shaded rectangles. That is,

$$\Delta(uv) = u(x+h)v(x+h) - u(x)v(x)$$
$$= u(x+h)\Delta v + v(x)\Delta u.$$

Division by h gives

$$\frac{\Delta(uv)}{h} = u(x+h)\frac{\Delta v}{h} + v(x)\frac{\Delta u}{h}.$$

$$\lim_{h\to 0}\frac{\Delta(uv)}{h} = \lim_{h\to 0}\left[u(x+h)\frac{\Delta v}{h} + v(x)\frac{\Delta u}{h}\right]$$

$$\Rightarrow \frac{d}{dx}(uv) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

Proof of the derivative product rule

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h} = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right]$$

$$= \lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}.$$

Reference:

Hass, J., Weir, M. D, and Thomas, G. B., University Calculus, Early Transcendentals, Second Edition, Pearson Education, Inc., 2012, P. 134.

Chain rule:

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