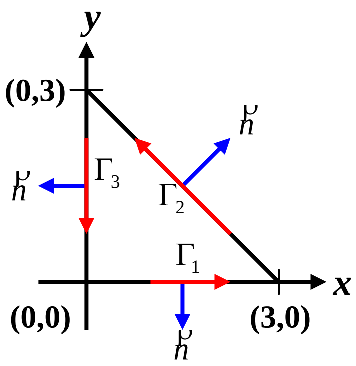
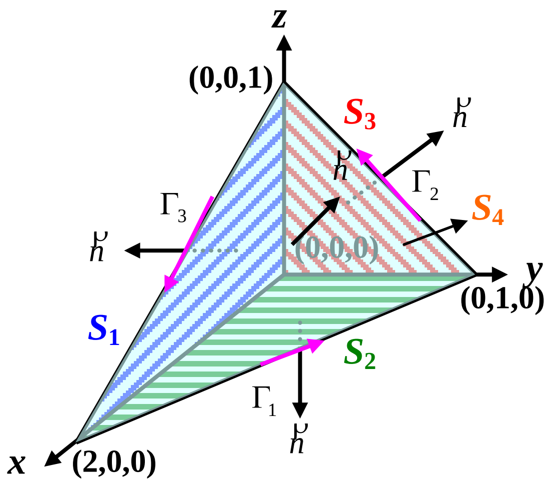
班級：______________學號：______________姓名：______________

國立海洋大學河海工程研究所 Gauss Stokes (2013)/NTOU-MSV





1. $\oint_C \vec{r}\cdot\vec{n}\,ds = \iint \nabla\cdot\vec{r}\,dA = ?$

   where $\vec{r} = (x, y)$

   $\vec{n}$: normal vector

   C: contour conntcting

   $(0,0) \rightarrow (3,0) \rightarrow (0,3) \rightarrow (0,0)$

   A: area of triangle

   ds: 弧長積分

2. $\oint_S \vec{r}\cdot\vec{n}\,dS = \iiint \nabla\cdot\vec{r}\,dV = ?$

   $\oint_s \vec{r}\cdot\vec{t}\,ds = \iint (\nabla\times\vec{r})\cdot\vec{n}\,dS = ?$

   where $\vec{r} = (x, y, z)$

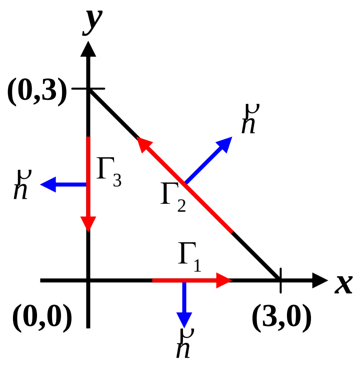
   $\vec{n}$: normal vector

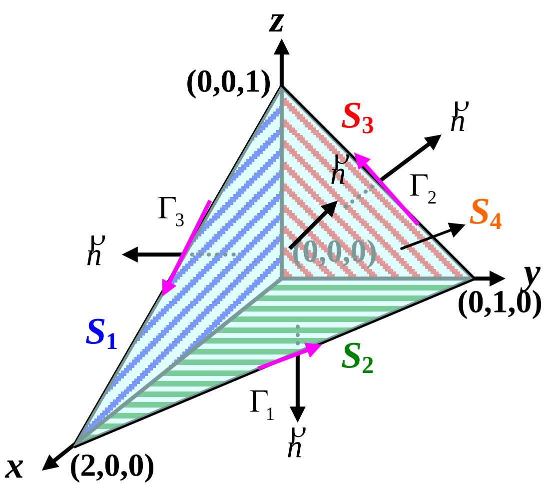
   S: surface of pyramid

   V: volume of pyramid

   dS: 表面積分

Ans:





1. $\oint_C \vec{F}\cdot\vec{n}\,ds = \int_{\Gamma_1}\vec{F}\cdot\vec{n}\,ds + \int_{\Gamma_2}\vec{F}\cdot\vec{n}\,ds + \int_{\Gamma_3}\vec{F}\cdot\vec{n}\,ds$

$$= \int_{\Gamma_2}(x,y)\cdot(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\,ds$$

$$= \int_0^{3\sqrt{2}}\frac{(x+y)}{\sqrt{2}}\,ds$$ (ps. 直線方程式: $x+y=3$)

$$= \int_0^{3\sqrt{2}}\frac{3}{\sqrt{2}}\,ds$$

$$= \frac{3}{\sqrt{2}}\cdot 3\sqrt{2} = 9$$

$$\iint \nabla\cdot\vec{F}\,dA = \iint (\frac{\partial}{\partial x},\frac{\partial}{\partial y})\cdot(x,y)\,dA$$

$$= \iint 2\,dA$$

$$= 2\cdot\frac{9}{2} = 9$$

2. $\oint_S \vec{F}\cdot\vec{n}\,dS = \int_{S_1}\vec{F}\cdot\vec{n}\,dS + \int_{S_2}\vec{F}\cdot\vec{n}\,dS + \int_{S_3}\vec{F}\cdot\vec{n}\,dS + \int_{S_4}\vec{F}\cdot\vec{n}\,dS$

$$= \int_{S_4}(x,y,z)\cdot(\frac{1}{3},\frac{2}{3},\frac{2}{3})\,dS$$

$$= \frac{2}{3}\int_{S_4}dS = 1$$

$$\iiint \nabla\cdot\vec{F}\,dV = \iiint (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\cdot(x,y,z)\,dV$$

$$= \iiint 3\,dV$$

$$= 3\cdot\frac{1}{3}(\frac{1}{2}\cdot 2\cdot 1)\cdot 1 = 1$$

ps.

$$\vec{n} = \frac{\begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix}}{\left\| \begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} \right\|} = \frac{1}{3}(i + 2j + 2k)$$

平面方程: $\frac{1}{3}(x-2) + \frac{2}{3}y + \frac{2}{3}z = 0 \Rightarrow \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z = \frac{2}{3}$

$A_{S_4} = \frac{3}{2}$

$$\oint_s \vec{r} \cdot \vec{t}\, ds = \oint_s (x,y,z) \cdot (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds})\ ds$$

$$= \int_{\Gamma_1} (x,y,z) \cdot (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) ds + \int_{\Gamma_2} (x,y,z) \cdot (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) ds$$

$$+ \int_{\Gamma_3} (x,y,z) \cdot (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) ds$$

$$= \int_{\Gamma_1} xdx + ydy + zdz + \int_{\Gamma_2} xdx + ydy + zdz + \int_{\Gamma_3} xdx + ydy + zdz$$

$$= \int_2^0 [x - \frac{1}{2}(1 - \frac{1}{2}x)]dx \ + \int_1^0 [y - (1-y)]dy \ + \int_2^0 [x - \frac{1}{2}(1 - \frac{1}{2}x)]dx$$

$$= \int_2^0 (\frac{5}{4}x - \frac{1}{2})dx \ + \int_1^0 (2y - 1)]dy \ + \int_0^2 (\frac{5}{4}x - \frac{1}{2})dx$$

$$= (\frac{5}{8}x^2 - \frac{1}{2}x)\Big|_2^0 + (y^2 - y)\Big|_1^0 + (\frac{5}{8}x^2 - \frac{1}{2}x)\Big|_0^2$$

$$= 0$$

$\Gamma_1 : x + 2y = 2 \quad \Rightarrow dx = -2dy$

$\Gamma_2 : y + z = 1 \quad \Rightarrow dy = -dz$

$\Gamma_3 : x + 2z = 2 \quad \Rightarrow dx = -2dz$

$$\iint (\nabla \times \vec{r}) \cdot \vec{n} dS = \iint (\nabla \times \vec{r}) \cdot \vec{n} dS = 0$$

$$\Theta\ \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

另解:

由 $\nabla\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0$ 得知 $\vec{r}$ 為一保守場

可知存在一保守場 $\phi(x,y,z)$

使得 $\nabla\phi=\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

所以可得 $\phi=\frac{1}{2}(x^2+y^2+z^2)$

$$\oint_s \vec{r}\cdot\vec{t}\,ds=\int_{\Gamma_1}\nabla\phi\cdot\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)ds+\int_{\Gamma_2}\nabla\phi\cdot\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)ds$$
$$+\int_{\Gamma_3}\nabla\phi\cdot\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)ds$$
$$=\int_{\Gamma_1}d\phi+\int_{\Gamma_2}d\phi+\int_{\Gamma_3}d\phi$$

$$\int_{\Gamma_1}d\phi=\int_{(2,0,0)}^{(0,1,0)}d\phi=\phi(0,1,0)-\phi(2,0,0)=\frac{1}{2}-\frac{4}{2}=-\frac{3}{2}$$

$$\int_{\Gamma_2}d\phi=\int_{(0,1,0)}^{(0,0,1)}d\phi=\phi(0,0,1)-\phi(0,1,0)=\frac{1}{2}-\frac{1}{2}=0$$

$$\int_{\Gamma_3}d\phi=\int_{(0,0,1)}^{(2,0,0)}d\phi=\phi(2,0,0)-\phi(0,0,1)=\frac{4}{2}-\frac{1}{2}=\frac{3}{2}$$

$$\therefore\oint_s \vec{r}\cdot\vec{t}\,ds=\int_{\Gamma_1}d\phi+\int_{\Gamma_2}d\phi+\int_{\Gamma_3}d\phi=-\frac{3}{2}+0+\frac{3}{2}=0$$