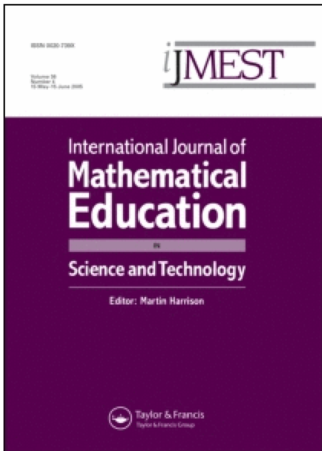

International Journal of Mathematical Education in Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713736815

The classical version of Stokes' theorem revisited

Steen Markvorsen [a]
[a] Department of Mathematics and LearningLab DTU, Technical University of Denmark, Lyngby, Denmark

First Published on: 04 June 2008

To cite this Article: Markvorsen, Steen (2008) 'The classical version of Stokes' theorem revisited', International Journal of Mathematical Education in Science and Technology,

To link to this article: DOI: 10.1080/00207390802091146
URL: http://dx.doi.org/10.1080/00207390802091146

International Journal of Mathematical Education in Science and Technology
2008, 1–10, iFirst

The classical version of Stokes' theorem revisited

Steen Markvorsen*

Department of Mathematics and LearningLab DTU, Technical University of Denmark, Lyngby, Denmark

(*Received 30 January 2007*)

Using only fairly simple and elementary considerations – essentially from first year undergraduate mathematics – we show how the classical Stokes' theorem for any given surface and vector field in $\mathbb{R}^3$ follows from an application of Gauss' divergence theorem to a suitable modification of the vector field in a tubular shell around the given surface. The two stated classical theorems are (like the fundamental theorem of calculus) nothing but shadows of the general version of Stokes' theorem for differential forms on manifolds. However, the main point in the present article is first, that this latter fact usually does not get within reach for students in first year calculus courses and second, that calculus textbooks in general only just hint at the correspondence alluded to above. Our proof that Stokes' theorem follows from Gauss' divergence theorem goes via a well-known and often used exercise, which simply relates the concepts of divergence and curl on the local differential level. The rest of this article uses only integration in 1, 2 and 3 variables together with a 'fattening' technique for surfaces and the inverse function theorem.

Keywords: Stokes' theorem; Gauss' divergence theorem; undergraduate mathematics; curriculum

2000 AMS Subject Classifications: Primary 26

1. Introduction

One of the most elegant and useful results concerning vector fields in $\mathbb{R}^3$, is the classical version of Stokes' theorem. It is one of those important results, which is so nicely moulded from analysis, calculus, geometry, and linear algebra that it forms a solid basis for and indeed an integral part of the final fireworks and climax of first year undergraduate mathematics education. At the same time Stokes' theorem points towards a wealth of deep applications in electromagnetism, in fluid dynamics and in mathematics itself, to mention but a few of the most significant fields of applications. Such results serve as indispensable bootstraps for university students *en masse* – be they students of engineering, of physics, of biology, of chemistry, of mathematics, etc.; see e.g. [1,2].

It is the purpose of this article to facilitate the presentation, i.e. the undergraduate teaching, of Stokes' theorem by suggesting and unfolding a proof, which shows that it is a direct consequence of Gauss' divergence theorem. In the process, there are also a few other useful insights and geometric observations to be (re)visited. The first pertinent

*Email: S.Markvorsen@mat.dtu.dk

ISSN 0020–739X print/ISSN 1464–5211 online

DOI: 10.1080/00207390802091146
http://www.informaworld.com

observation is the following quotation from [3] concerning the history of Stokes' theorem (see also [4] p. 790):

> The history of Stokes' Theorem is clear but very complicated. It was first given by Stokes without proof – as was necessary – since it was given as an examination question for the Smith's Prize Examination of that year [at Cambridge in 1854]! Among the candidates for the prize was Maxwell, who later traced to Stokes the origin of the theorem, which by 1870 was frequently used. On this see George Gabriel Stokes, *Mathematical and Physical Papers*, vol. V (Cambridge, England, 1905), 320–321. See also the important historical footnote which indicates that Kelvin in a letter of 1850 was the first who actually stated the theorem, although others as Ampère had employed "the same kind of analysis ... in particular cases."
>
> M. J. Crowe, [3] p. 147.

Outline of this article: After a presentation of Gauss' theorem and Stokes' theorem in Section 2 we recall (in Section 3) the connection between curl and divergence which was alluded to in the abstract. In the following sections, we then set up the notation and the results needed in order to make this work reasonably self contained on the level of first year undergraduate mathematics. The main goal is to relate integration over the shell extension of a given surface with the integration along the boundary surfaces of this extension. The proof of Stokes' theorem is finally completed in Section 9.

2. Fundamental theorems of calculus

Gauss' divergence theorem is of the same calibre as Stokes' theorem. They are both members of a family of results which are concerned with 'pushing the integration to the boundary'. The eldest member of this family is the following:

Theorem 2.1 (Fundamental theorem of calculus)**:** *Let f be a continuous function on $\mathbb{R}$. Then the function*

$$A(x) = \int_0^x f(u)\mathrm{d}u$$

is differentiable with

$$A'(x) = f(x), \tag{2.1}$$

and moreover, if $F(x)$ is any (other) function satisfying $F'(x) = f(x)$, then

$$\int_a^b f(u)\mathrm{d}u = F(b) - F(a). \tag{2.2}$$

The message of this theorem is that two fundamental problems – that of finding a function whose derivative is a given function and that of finding the average of a given function – have a common solution. It is also the first result which displays – in Equation (2.2) – the astounding success of 'pushing the integration to the boundary'. Compare with the main 'actors' of the present article, Theorems 2.3 and 2.4 below.

The divergence theorem is not – conceptually speaking – 'far' from the fundamental theorem of calculus. Most textbook proofs of the divergence theorem covers only the special setting of a domain whose boundary consists of the graphs of two functions, each of two variables. This enables in fact a direct proof in this special case via Theorem 2.1, see [5] pp. 1058–1059. Stokes' theorem is a little hard to grasp, even locally, but follows also in the corresponding setting (for graph surfaces) from Gauss' theorem for planar domains, see [5] pp. 1065–1066.

This approach suggests indirectly that the full classical Stokes' theorem (for general surfaces) should follow directly from Gauss' divergence theorem (for general domains). The main part of the present article will be devoted to a proof following this idea.

The most compact as well as the most general form of Stokes' theorem reads as follows (see e.g. [6] p. 353, [7], [8] pp. 60 ff., [9] p. 124):

Theorem 2.2 (Stokes' theorem, general version): *Let ω denote a differential $(k-1)$-form on a compact orientable manifold Ω^k. Suppose that Ω has a smooth and compact boundary $\partial\Omega$ with the induced orientation, and let $\mathrm{d}\omega$ denote the differential of ω. Then*

$$\int_{\Omega} \mathrm{d}\omega = \int_{\partial\Omega} \omega. \tag{2.3}$$

As corollaries of this statement we have both Gauss' divergence theorem for domains in $\mathbb{R}^3$, and Stokes' theorem for surfaces in $\mathbb{R}^3$ (see, e.g. [6] pp. 319–320). As mentioned above it is these latter theorems – not the general version of Stokes' theorem – that will be the main concern of this article. Here are the statements:

Theorem 2.3 (Gauss' divergence theorem): *Let Ω denote a compact domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$ and outward pointing unit normal vector field $\mathbf{n}_{\partial\Omega}$ on $\partial\Omega$. Let $\mathbf{V}$ be a vector field in $\mathbb{R}^3$. Then*

$$\int_{\Omega} \mathrm{div}(\mathbf{V})\mathrm{d}\mu = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}_{\partial\Omega}\, \mathrm{d}\nu. \tag{2.4}$$

Theorem 2.4 (Stokes' theorem, classical version): *Let F denote a compact, orientable, regular and smooth surface with piecewise smooth boundary ∂F and unit normal vector field $\mathbf{n}_F$. Let $\mathbf{V}$ be a vector field in $\mathbb{R}^3$. Then*

$$\int_{F} \mathrm{curl}(\mathbf{V}) \cdot \mathbf{n}_F\, \mathrm{d}\mu = \int_{\partial F} \mathbf{V} \cdot \mathbf{e}_{\partial F}\, \mathrm{d}\sigma. \tag{2.5}$$

When calculating the right-hand side, i.e. the tangential curve integral (the circulation of $\mathbf{V}$) along the boundary ∂F, the orientation $\mathbf{e}_{\partial F}$ of the boundary must be chosen so that the cross product $\mathbf{e}_{\partial F} \times \mathbf{n}_F$ at the boundary points away from the surface.

3. A bridge between divergence and curl

We begin by stating a connection between divergence and curl.

Observation 3.1 (Exercise): Let $\mathbf{V}(x, y, z)$ and $\mathbf{W}(x, y, z)$ denote two smooth vector fields in $\mathbb{R}^3$. Then, the following identity holds true:

$$\mathrm{div}(\mathbf{V} \times \mathbf{W}) = \mathrm{curl}(\mathbf{V}) \cdot \mathbf{W} - \mathbf{V} \cdot \mathrm{curl}(\mathbf{W}). \tag{3.1}$$

In particular, if $\mathbf{W}$ is a gradient field for some smooth function $\psi(x, y, z)$ in $\mathbb{R}^3$, i.e. $\mathbf{W} = \mathrm{grad}(\psi)$, we get from $\mathrm{curl}(\mathrm{grad}(\psi)) = 0$:

$$\mathrm{div}(\mathbf{V} \times \mathrm{grad}(\psi)) = \mathrm{curl}(\mathbf{V}) \cdot \mathrm{grad}(\psi). \tag{3.2}$$

Using Gauss' divergence theorem we 'lift' this connection to the integral level as follows:

Theorem 3.2: *Let $\psi(x, y, z)$ denote a smooth function in $\mathbb{R}^3$ and let $\mathbf{V}(x, y, z)$ be a vector field. Let Ω denote a compact domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$ and outward pointing unit normal vector field $\mathbf{n}_{\partial\Omega}$ on $\partial\Omega$. Then we have the following*

$$\int_{\Omega} \operatorname{div}(\mathbf{V} \times \operatorname{grad}(\psi))\mathrm{d}\mu = \int_{\partial\Omega} (\mathbf{V} \times \operatorname{grad}(\psi)) \cdot \mathbf{n}_{\partial\Omega}\, \mathrm{d}\nu. \tag{3.3}$$

Using Equation (3.2), *we also have*

$$\int_{\Omega} \operatorname{curl}(\mathbf{V}) \cdot \operatorname{grad}(\psi)\mathrm{d}\mu = \int_{\partial\Omega} (\mathbf{n}_{\partial\Omega} \times \mathbf{V}) \cdot \operatorname{grad}(\psi)\mathrm{d}\nu. \tag{3.4}$$

In particular, we get the total rotation vector (the so-called total 'vorticity vector' of fluid dynamics) of the vector field $\mathbf{V}$ in Ω:

Corollary 3.3:

$$\int_{\Omega} \operatorname{curl}(\mathbf{V})\mathrm{d}\mu = \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \times \mathbf{V}\, \mathrm{d}\nu.$$

Proof: This follows directly from Equation (3.4) by choosing, in turn, $\psi(x, y, z) = x$, $\psi(x, y, z) = y$, and $\psi(x, y, z) = z$, so that $\operatorname{grad}(\psi)$ is successively one of the respective constant vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. □

4. The surface, the boundary and the normal field

We parametrize a given surface F by a smooth regular map $\mathbf{r}$ from a compact domain D (with boundary ∂D) in the (u, v)-plane into $\mathbb{R}^3$:

$$F : \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^3, \quad (u, v) \in D \subset \mathbb{R}^2,$$

where $x(u, v)$, $y(u, v)$, and $z(u, v)$ are smooth functions of the parameters u and v.

Example 4.1: The Costa surface in Figure 1 is obtained by a highly non-trivial parametric deformation of a disk D from which 4 smaller disks have been removed in the (u, v)-plane $\mathbb{R}^2$. Two of the 5 boundary components are identified by the map, so that the surface has the topology of a thrice punctured torus. Details on the construction of Costa's minimal surface can be found in e.g. [10].

The connected components of the boundary ∂D are either pairwise identified by $\mathbf{r}$ or mapped onto the components of ∂F, respectively. We assume, that $\mathbf{r}$ is everywhere bijective except at those components of ∂D which are identified by the map. In cases like Costa's surface – as shown in Figure 1 – we have several boundary components. They contribute additively and individually to the circulation integral on the right-hand side of Stokes' theorem. Those boundary components of ∂D which are identified by $\mathbf{r}$ do not contribute to ∂F. They do not contribute to the Stokes circulation integral either because the relevant integrals cancel each other away. For ease of presentation and without lack of generality, we therefore assume that D is simply

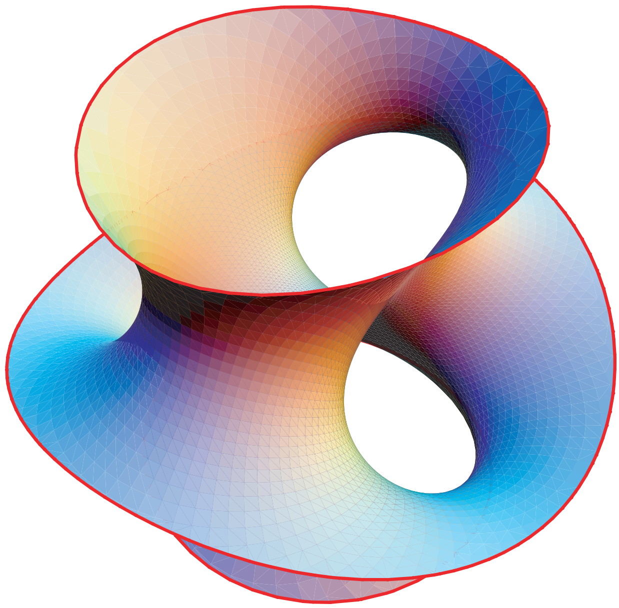

Figure 1. Costa's minimal surface.

connected with only one connected boundary component ∂D which is mapped onto ∂F via the map $\mathbf{r}$.

A given single boundary component ∂D is parametrized as follows in the (u, v)-plane:

$$\partial D : \mathbf{d}(\theta) = (u(\theta), v(\theta)) \in \partial D \subset \mathbb{R}^2, \quad \theta \in I \subset \mathbb{R},$$

where $u(\theta)$ and $v(\theta)$ are piecewise smooth functions of θ. The boundary of F is then

$$\partial F : \mathbf{b}(\theta) = \mathbf{r}(\mathbf{d}(\theta)) = \mathbf{r}(u(\theta), v(\theta)) \in \mathbb{R}^3.$$

The Jacobians of the maps $\mathbf{r}$ and $\mathbf{b}$ are, respectively:

$$\mathrm{Jacobi}_{\mathbf{r}}(u, v) = \|\mathbf{r}'_u \times \mathbf{r}'_v\|, \quad \text{and} \tag{4.1}$$

$$\mathrm{Jacobi}_{\mathbf{b}}(\theta) = \|\mathbf{b}'_\theta\|. \tag{4.2}$$

The regularity of $\mathbf{r}$ is expressed by

$$\mathrm{Jacobi}_{\mathbf{r}}(u, v) > 0 \quad \text{for all} \quad (u, v) \in D.$$

This implies, in particular, that there is a well-defined unit normal vector $\mathbf{n}_F = \mathbf{n}(u, v)$ at each point of F:

$$\mathbf{n}(u, v) = \frac{\mathbf{r}'_u \times \mathbf{r}'_v}{\|\mathbf{r}'_u \times \mathbf{r}'_v\|} \quad \text{for all} \quad (u, v) \in D. \tag{4.3}$$

5. The shell fattening and a nice gradient

We define the *tubular shell fattening of* F (of thickness t) as the following parametrized domain in $\mathbb{R}^3$:

$$\Omega_t : \mathbf{R}(u, v, w) = \mathbf{r}(u, v) + w\mathbf{n}(u, v), \quad (u, v) \in D, \quad w \in [0, t]. \tag{5.1}$$

In particular, the *surface* F is then the base surface of the shell and is obtained by restricting $\mathbf{R}$ to D (where $w = 0$):

$$F_0 = F : \mathbf{r}(u, v) = \mathbf{R}(u, v, 0), \quad (u, v) \in D.$$

Similarly, for $w = t$ we get the top surface F_t of the shell. It is parametrized by $\mathbf{R}(u, v, t)$, for $(u, v) \in D$.

The Jacobian of the map $\mathbf{R}$ is

$$\begin{aligned} \mathrm{Jacobi}_{\mathbf{R}}(u, v, w) &= |(\mathbf{R}'_u \times \mathbf{R}'_v) \cdot \mathbf{R}'_w| \\ &= |((\mathbf{r}'_u + w\,\mathbf{n}'_u) \times (\mathbf{r}'_v + w\,\mathbf{n}'_v)) \cdot \mathbf{n}_F|, \end{aligned} \tag{5.2}$$

so that, since $\mathbf{n}_F$ is a unit vector field parallel to $\mathbf{r}'_u \times \mathbf{r}'_v$ along F, we get in particular (for $w = 0$):

$$\begin{aligned} \mathrm{Jacobi}_{\mathbf{R}}(u, v, 0) &= |(\mathbf{r}'_u \times \mathbf{r}'_v) \cdot \mathbf{n}_F| \\ &= \|\mathbf{r}'_u \times \mathbf{r}'_v\| \\ &= \mathrm{Jacobi}_{\mathbf{r}}(u, v) > 0. \end{aligned} \tag{5.3}$$

The map $\mathbf{R}$ is regular and bijective on $D \times [0, t]$ – provided t is sufficiently small. Indeed, since $\mathrm{Jacobi}_{\mathbf{R}}(u, v, 0) > 0$, this claim follows from the continuity of $\mathrm{Jacobi}_{\mathbf{R}}(u, v, w)$ and the compactness of D.

The value of w considered as a function in $\Omega_t \subset \mathbb{R}^3$ is a smooth function of the coordinates (x, y, z). However intuitively reasonable this claim may seem, the precise argument goes via the inverse function theorem, which we state here for completeness – in its global form, without proof:

Theorem 5.1: *Let Q denote an open set in $\mathbb{R}^n$ and let $\mathbf{f} : Q \to \mathbb{R}^n$ denote a smooth bijective map with* $\mathrm{Jacobi}_{\mathbf{f}}(\mathbf{x}) > 0$ *for all $\mathbf{x} \in Q$. Then, the inverse map $\mathbf{f}^{-1} : \mathbf{f}(Q) \to Q$ is also smooth with* $\mathrm{Jacobi}_{\mathbf{f}^{-1}}(\mathbf{y}) > 0$ *for all $\mathbf{y} \in \mathbf{f}(Q)$.*

Hence, when t is sufficiently small, w is a smooth function of (x, y, z); let us call it $h(x, y, z)$, $(x, y, z) \in \Omega_t$. This function then has a non-vanishing gradient, $\mathrm{grad}(h)(x, y, z)$, which is orthogonal to the level surfaces of h. In particular, $\mathrm{grad}(h)$ is orthogonal to the top surface F_t of the shell Ω_t, where $h = t$ and it is orthogonal to the base surface $F_0 = F$, where $h = 0$.

In fact, at the base surface, the field $\mathrm{grad}(h)$ is precisely equal to the unit normal vector field $\mathbf{n}_F$. To see this we only need to show that it has unit length: Let (u_0, v_0) denote a given point in D and consider the restriction of h to the straight line $\mathbf{r}(u_0, v_0) + w\mathbf{n}(u_0, v_0)$, where $w \in [0, t]$. Let us denote $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ and $\mathbf{n}_0 = \mathbf{n}(u_0, v_0)$. The chain rule then gives

$$\begin{aligned} 1 &= \left| \frac{\mathrm{d}}{\mathrm{d}w} h(\mathbf{r}_0 + w\,\mathbf{n}_0) \right| \\ &= |\,\mathbf{n}_0 \cdot \mathrm{grad}(h)(\mathbf{r}_0 + w\,\mathbf{n}_0)\,| \\ &= \|\mathrm{grad}(h)(\mathbf{r}_0 + w\,\mathbf{n}_0)\|, \end{aligned} \tag{5.4}$$

so that, at the surface F, for $w = 0$, we have $\|\mathrm{grad}(h)(\mathbf{r}_0)\| = 1$ and therefore in total, as claimed above:

$$\mathrm{grad}(h)_{|_F} = \mathbf{n}_F, \tag{5.5}$$

Remark 5.2: The function $h(x, y, z)$ is in fact the *Euclidean distance* from the point (x, y, z) in Ω_t to the surface F.

6. Integration in the shell

For any given smooth function $f(x, y, z)$ defined in Ω_t, the integral of f over that domain is:

$$\int_{\Omega_t} f\,\mathrm{d}\mu = \int_0^t \left(\int_D f(\mathbf{R}(u, v, w))\mathrm{Jacobi}_{\mathbf{R}}(u, v, w)\mathrm{d}u\,\mathrm{d}v \right)\mathrm{d}w. \tag{6.1}$$

The derivative of this integral with respect to the thickness t of the shell Ω_t is, at $t = 0$, the surface integral over F:

Lemma 6.1:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|_{t=0}} \int_{\Omega_t} f\,\mathrm{d}\mu = \int_F f\,\mathrm{d}\nu. \tag{6.2}$$

Proof: This follows directly from the fundamental theorem of calculus, Theorem 2.1, Equation (2.1):

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|_{t=0}} \int_{\Omega_t} f\,\mathrm{d}\mu &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|_{t=0}} \int_0^t \left(\int_D f(\mathbf{R}(u, v, w))\mathrm{Jacobi}_{\mathbf{R}}(u, v, w)\mathrm{d}u\,\mathrm{d}v \right)\mathrm{d}w \\
&= \int_D f(\mathbf{R}(u, v, 0))\mathrm{Jacobi}_{\mathbf{R}}(u, v, 0)\mathrm{d}u\,\mathrm{d}v \\
&= \int_D f(\mathbf{r}(u, v))\mathrm{Jacobi}_{\mathbf{r}}(u, v)\mathrm{d}u\,\mathrm{d}v \\
&= \int_F f\,\mathrm{d}\nu.
\end{aligned} \tag{6.3}$$

□

7. The wall

The shell Ω_t has a boundary $\partial\Omega_t$ which consists of the top level surface F_t, the base level surface $F = F_0$ and a 'cylindrical wall' surface W_t of height t (Figure 2). This latter component of the boundary is simply obtained by restricting the map R to $\partial D \times [0, t]$ as follows:

$$\begin{aligned}
W_t : \mathbf{B}(\theta, w) &= \mathbf{R}(u(\theta), v(\theta), w) \\
&= \mathbf{r}(u(\theta), v(\theta)) + w\,\mathbf{n}(u(\theta), v(\theta)) \\
&= \mathbf{b}(\theta) + w\,\mathbf{n}(\mathbf{d}(\theta)), \quad \theta \in I,\ w \in [0, t].
\end{aligned}$$

The Jacobian of this map is thus

$$\begin{aligned} \mathrm{Jacobi}_{\mathbf{B}}(\theta, w) &= \|\mathbf{B}'_\theta \times \mathbf{B}'_w\| \\ &= \|(\mathbf{b}'_\theta + w\,(\mathbf{n} \circ \mathbf{d})'_\theta) \times \mathbf{n}_F\|, \end{aligned} \tag{7.1}$$

so that, since $\mathbf{n}_F$ is a unit normal to the surface F and hence also to the boundary ∂F (parametrized by $\mathbf{b}$), we get in particular:

$$\mathrm{Jacobi}_{\mathbf{B}}(\theta, 0) = \|\mathbf{b}'_\theta \times \mathbf{n}_F\| = \|\mathbf{b}'_\theta\| = \mathrm{Jacobi}_{\mathbf{b}}(\theta). \tag{7.2}$$

8. Integration along the wall

For any given smooth function $g(x, y, z)$ defined on W_t the integral of g over that surface is

$$\int_{W_t} g\,\mathrm{d}\nu = \int_0^t \left(\int_I g(\mathbf{B}(\theta, w))\,\mathrm{Jacobi}_{\mathbf{B}}(\theta, w)\mathrm{d}\theta \right) \mathrm{d}w. \tag{8.1}$$

The derivative of this integral with respect to the height t of the wall W_t is, at $t = 0$, the line integral over ∂F:

Lemma 8.1:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|_{t=0}} \int_{W_t} g\,\mathrm{d}\nu = \int_{\partial F} g\,\mathrm{d}\sigma. \tag{8.2}$$

Proof: This follows again from the fundamental theorem of calculus, Theorem 2.1, Equation (2.1):

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|_{t=0}} \int_{W_t} g\,\mathrm{d}\nu &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|_{t=0}} \int_0^t \left(\int_I g(\mathbf{B}(\theta, w))\mathrm{Jacobi}_{\mathbf{B}}(\theta, w)\mathrm{d}\theta \right) \mathrm{d}w \\ &= \int_I g(\mathbf{B}(\theta, 0))\mathrm{Jacobi}_{\mathbf{B}}(\theta, 0)\mathrm{d}\theta \\ &= \int_I g(\mathbf{b}(\theta))\mathrm{Jacobi}_{\mathbf{b}}(\theta)\mathrm{d}\theta \\ &= \int_{\partial F} g\,\mathrm{d}\sigma. \end{aligned} \tag{8.3}$$

□

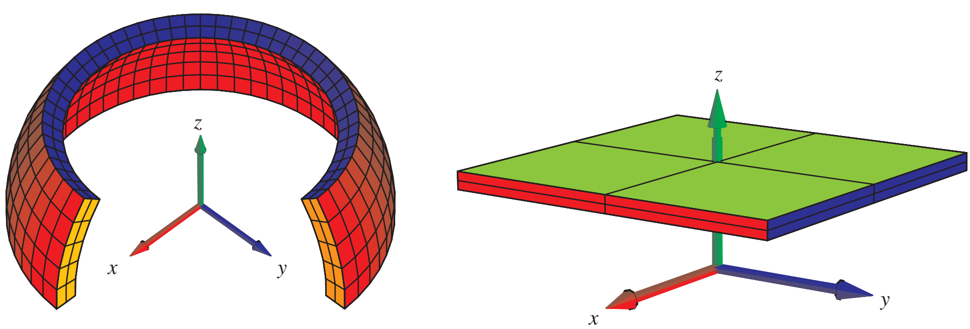

Figure 2. Two shells obtained as small fattenings of a piece of a sphere and a square, respectively.

9. Proof of Stokes' theorem for surfaces

We are now ready to prove Theorem 2.4.

Proof: Using the function $h(x, y, z)$ from Section 5 in place of the function $\psi(x, y, z)$ in Theorem 3.2, Equation (3.4) for the domain $\Omega = \Omega_t$ we get:

$$\begin{aligned}\int_{\Omega_t} \mathrm{curl}(\mathbf{V}) \cdot \mathrm{grad}(h) \mathrm{d}\mu &= \int_{\partial\Omega_t} \left(\mathbf{n}_{\partial\Omega_t} \times \mathbf{V}\right) \cdot \mathrm{grad}(h)\, \mathrm{d}\nu \\ &= \int_{F_t} \left(\mathbf{n}_{F_t} \times \mathbf{V}\right) \cdot \mathrm{grad}(h)\, \mathrm{d}\nu - \int_{F_0} \left(\mathbf{n}_{F_0} \times \mathbf{V}\right) \cdot \mathrm{grad}(h)\, \mathrm{d}\nu \\ &\quad + \int_{W_t} \left(\mathbf{n}_{W_t} \times \mathbf{V}\right) \cdot \mathrm{grad}(h)\, \mathrm{d}\nu. \end{aligned} \tag{9.1}$$

But in Equation (9.1) we have

$$\int_{F_t} \left(\mathbf{n}_{F_t} \times \mathbf{V}\right) \cdot \mathrm{grad}(h)\, \mathrm{d}\nu = 0$$

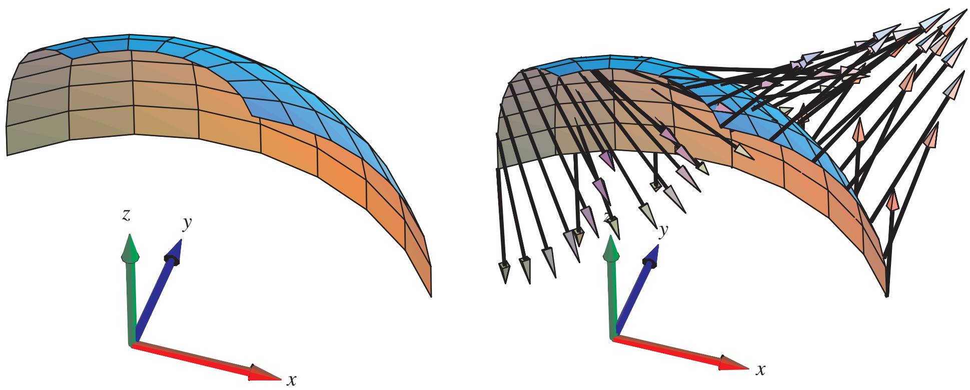

Figure 3. A piece of a sphere and the curl vector field curl(**V**) of the field $\mathbf{V}(x, y, z) = (z^2\, x, x^2\, y, y^2\, z)$. The field **V** itself is not shown.

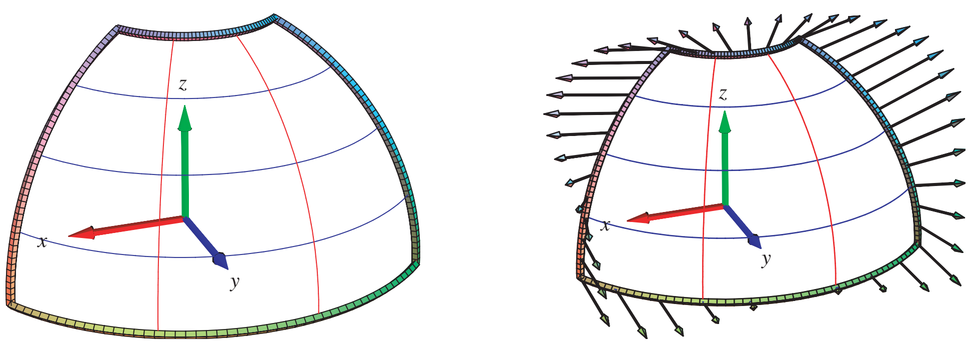

Figure 4. According to Stokes' theorem, the flux through the surface shown in Figure 3 is equal to the circulation of the vector field **V** along the boundary curve of the surface. The field **V** is shown here along that boundary.

and

$$\int_{F_0} \left(\mathbf{n}_{F_0} \times \mathbf{V}\right) \cdot \operatorname{grad}(h)\, \mathrm{d}\nu = 0, \tag{9.2}$$

because grad(h) is orthogonal to both of the surfaces F_t and F_0 so that grad(h) is proportional to $\mathbf{n}_{F_t}$ and $\mathbf{n}_{F_0}$ at the respective surfaces.

We observe that at $\partial F \subset W_t$ we have $\mathbf{n}_{W_t} = \mathbf{e}_{\partial F} \times \mathbf{n}_F$ and hence $\mathbf{e}_{\partial F} = \mathbf{n}_F \times \mathbf{n}_{W_t}$ according to the rule in Theorem 2.4, which defines the orientation of ∂F. Taking derivatives in Equation (9.1) with respect to t at $t = 0$ then gives:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|t=0} \int_{\Omega_t} \operatorname{curl}(\mathbf{V}) \cdot \operatorname{grad}(h) \mathrm{d}\mu = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|t=0} \int_{W_t} \left(\mathbf{n}_{W_t} \times \mathbf{V}\right) \cdot \operatorname{grad}(h)\, \mathrm{d}\nu, \tag{9.3}$$

so that – by the virtues of Equations (6.2) and (8.2) – we finally get

$$\begin{aligned} \int_F \operatorname{curl}(\mathbf{V}) \cdot \mathbf{n}_F\, \mathrm{d}\nu &= \int_{\partial F} \left(\mathbf{n}_{W_t} \times \mathbf{V}\right) \cdot \mathbf{n}_F\, \mathrm{d}\sigma \\ &= \int_{\partial F} \mathbf{V} \cdot \left(\mathbf{n}_F \times \mathbf{n}_{W_t}\right) \mathrm{d}\sigma \\ &= \int_{\partial F} \mathbf{V} \cdot \mathbf{e}_{\partial F}\, \mathrm{d}\sigma, \end{aligned} \tag{9.4}$$

which finishes the proof of the theorem. □

We refer to Figures 3 and 4 which display the setting for a concrete application of Stokes' theorem to a given vector field and a given portion of a sphere.

Acknowledgements

This work was carried out while the author was part time affiliated with the LearningLab DTU. The author would like to thank the staff and colleagues at the LearningLab DTU for their kind hospitality, rewarding and inspiring discussions on curriculum, teaching, and learning, and for technical and financial support.

References

[1] R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. II, Addison Wesley, San Francisco, 1965.

[2] J.A. Shercliff, *Vector fields; Vector Analysis Developed Through its Application to Engineering and Physics*, Cambridge University Press, Cambridge, 1977.

[3] M.J. Crowe, *A History of Vector Analysis*, Dover, New York, 1994.

[4] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Vol. 2, Oxford University Press, New York, 1972.

[5] C.H. Edwards and D.E. Penney, *Calculus*, 6th ed., Prentice Hall, New Jersey, 2002.

[6] J.R. Munkres, *Analysis on Manifolds*, Addison Wesley, California, 1991.

[7] H. Grunsky, *The General Stokes' Theorem*, Pitman, Boston, 1983.

[8] M.P. doCarmo, *Differential Forms and Applications*, Springer, Berlin, 1994.

[9] M. Spivak, in *Calculus on Manifolds*, W.A. Benjamin, ed., Inc., California, 1965.

[10] H. Ferguson, A. Gray, and S. Markvorsen, *Costa's minimal surface via Mathematica*, Math. Educat. Res. 5 (1996), pp. 5–10.