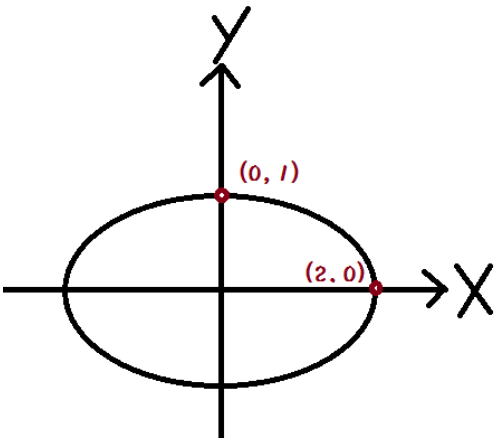

$y=\sqrt{1-\frac{x^2}{4}}$ 之橢圓求(0,1),(2,0)曲率

1.用卡式座標求解 $\rho=\frac{(1+y'^2)^{\frac{3}{2}}}{y''}$

2.以參數式用極座標求解 $\rho=\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}$

1.

$$y=\sqrt{1-\frac{x^2}{4}},y'=\frac{1}{2}(1-\frac{x^2}{4})^{-\frac{1}{2}}(-\frac{x}{2})=(-\frac{x}{4})(1-\frac{x^2}{4})^{-\frac{1}{2}},$$

$$y''=(-\frac{1}{4})(1-\frac{x^2}{4})^{-\frac{1}{2}}-(-\frac{x}{4})(\frac{-1}{2})(1-\frac{x^2}{4})^{-\frac{3}{2}}(-\frac{x}{2})=-\frac{1}{4}(1-\frac{x^2}{4})^{-\frac{1}{2}}-\frac{x^2}{16}(1-\frac{x^2}{4})^{-\frac{3}{2}}$$

$$1+y'^2=1+(\frac{x^2}{16})(\frac{4}{4-x^2})=1+(\frac{x^2}{4})(\frac{1}{4-x^2})=\frac{4(4-x^2)+x^2}{4(4-x^2)}=\frac{16-3x^2}{4(4-x^2)}$$

$$y''=-\frac{1}{4}(1-\frac{x^2}{4})^{-\frac{1}{2}}-\frac{x^2}{16}(1-\frac{x^2}{4})^{-\frac{3}{2}}=-\frac{1}{4}(\frac{4}{4-x^2})^{\frac{1}{2}}-\frac{x^2}{16}(\frac{4}{4-x^2})^{\frac{3}{2}}=-\frac{1}{4}\times4^{\frac{1}{2}}(\frac{1}{4-x^2})^{\frac{1}{2}}-\frac{x^2}{16}\times4^{\frac{3}{2}}(\frac{1}{4-x^2})^{\frac{3}{2}}$$

$$=-\frac{1}{2}(\frac{1}{4-x^2})^{\frac{1}{2}}-\frac{x^2}{2}\times(\frac{1}{4-x^2})^{\frac{3}{2}}$$

$$\rho=\frac{(1+y'^2)^{\frac{3}{2}}}{y''}=\frac{(\frac{16-3x^2}{4(4-x^2)})^{\frac{3}{2}}}{-\frac{1}{2}(\frac{1}{4-x^2})^{\frac{1}{2}}-\frac{x^2}{2}\times(\frac{1}{4-x^2})^{\frac{3}{2}}}=\frac{(\frac{16-3x^2}{4})^{\frac{3}{2}}\times(\frac{1}{4-x^2})^{\frac{3}{2}}}{-\frac{1}{2}(\frac{1}{4-x^2})^{\frac{1}{2}}-\frac{x^2}{2}\times(\frac{1}{4-x^2})^{\frac{3}{2}}}=\frac{(\frac{16-3x^2}{4})^{\frac{3}{2}}\times(\frac{1}{4-x^2})^{\frac{2}{2}}}{-\frac{1}{2}-\frac{x^2}{2}\times(\frac{1}{4-x^2})^{\frac{2}{2}}}$$

$$=\frac{(\frac{16-3x^2}{4})^{\frac{3}{2}}\times(\frac{1}{4-x^2})}{\frac{-(4-x^2)-x^2}{2(4-x^2)}}=\frac{(\frac{16-3x^2}{4})^{\frac{3}{2}}\times(\frac{1}{4-x^2})}{\frac{-4+x^2-x^2}{2(4-x^2)}}=\frac{(\frac{16-3x^2}{4})^{\frac{3}{2}}\times(\frac{1}{4-x^2})}{\frac{-4}{2(4-x^2)}}=\frac{(\frac{16-3x^2}{4})^{\frac{3}{2}}\times(\frac{1}{4-x^2})}{\frac{-2}{(4-x^2)}}$$

$$=\frac{(\frac{16-3x^2}{4})^{\frac{3}{2}}}{-2}=\frac{(16-3x^2)^{\frac{3}{2}}\times(\frac{1}{4})^{\frac{3}{2}}}{-2}=\frac{(16-3x^2)^{\frac{3}{2}}\times\frac{1}{8}}{-2}=\frac{(16-3x^2)^{\frac{3}{2}}}{-16}$$

當 $x=2\Rightarrow\rho_1=\frac{(16-3(2)^2)^{\frac{3}{2}}}{-16}=\frac{(16-12)^{\frac{3}{2}}}{-16}=\frac{(4)^{\frac{3}{2}}}{-16}=\frac{8}{-16}=\frac{1}{-2}$

當 $x=0\Rightarrow\rho_2=\frac{(16-3(0)^2)^{\frac{3}{2}}}{-16}=\frac{(16)^{\frac{3}{2}}}{-16}=\frac{64}{-16}$=-4

2.

$x = 2\cos t, \dot{x} = -2\sin t, \ddot{x} = -2\cos t$

$y = \sin t, \dot{y} = \cos t, \ddot{y} = -\sin t$

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} = \frac{(4\sin^2 t + \cos^2 t)^{\frac{3}{2}}}{2\sin^2 t + 2\cos^2 t} = \frac{(4\sin^2 t + \cos^2 t)^{\frac{3}{2}}}{2}$$

當 $t = 0 \Rightarrow \rho_1 = \frac{(4\sin^2 0 + \cos^2 0)^{\frac{3}{2}}}{2} = \frac{1}{2}$

當 $t = \frac{\pi}{2} \Rightarrow \rho_1 = \frac{(4\sin^2 \frac{\pi}{2} + \cos^2 \frac{\pi}{2})^{\frac{3}{2}}}{2} = \frac{4^{\frac{3}{2}}}{2} = \frac{8}{2} = 4$

結論:在求橢圓之曲率半徑時，用極座標求解計算得過程較卡式座標快速，所以我會比較喜歡用極座標設參數式來求解。