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FOURIER SERIES AND BOUNDARY VALUE PROBLEMS

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The continuous function

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{when } x \neq 0, \\ 0 & \text{when } x = 0 \end{cases} \tag{6}$$

illustrates the distinction between one-sided derivatives and one-sided limits of derivatives. Here $f'_R(0) = f'_L(0) = 0$, while the one-sided limits $f'(0+)$ and $f'(0-)$ do not exist. The verification of this is left as a problem.

40. Preliminary Theory

We begin our discussion of the convergence of Fourier series with two preliminary theorems, or lemmas. The first is often referred to as the *Riemann-Lebesgue lemma*, and we present it in somewhat greater generality than we actually need in order that it can be used as well in Chap. 7, where the convergence of Fourier integrals is treated, and also in Chap. 8.

Lemma 1. *If a function $G(u)$ is piecewise continuous on an interval $(0,c)$, then*

$$\lim_{r\to\infty} \int_0^c G(u) \sin ru\, du = \lim_{r\to\infty} \int_0^c G(u) \cos ru\, du = 0. \tag{1}$$

是相同

We shall verify only the first limit here and leave verification of the second, which is similar, to the problems.

To verify the first limit, it is sufficient to show that if $G(u)$ is continuous at each point of an interval $a \le u \le b$, then

$$\lim_{r\to\infty} \int_a^b G(u) \sin ru\, du = 0. \tag{2}$$

For, in view of the discussion of integrals of piecewise continuous functions in Sec. 24, the integral in the first of limits (1) can be expressed as the sum of a finite number of integrals of the type appearing in equation (2).

Assuming, then, that $G(u)$ is continuous on the closed bounded interval $a \le u \le b$, we note that it must also be *uniformly* continuous there. That is, for each positive number ε there exists a positive number δ such that $|G(u) - G(v)| < \varepsilon$ whenever u and v lie in the interval and satisfy the inequality $|u - v| < \delta$.† Writing

$$\varepsilon = \frac{\varepsilon_0}{2(b-a)}$$

† See, for example, Taylor and Mann (1972, pp. 558–561) or Buck (1978, Sec. 2.3), listed in the Bibliography.

where ε_0 is an arbitrary positive number, we are thus assured that there is a positive number δ such that

$$|G(u) - G(v)| < \frac{\varepsilon_0}{2(b-a)} \quad \text{whenever } |u - v| < \delta. \tag{3}$$

To obtain the limit (2), divide the interval $a \le u \le b$ into N subintervals of equal length $(b-a)/N$ by means of the points $a = u_0, u_1, u_2, \ldots, u_N = b$, where $u_0 < u_1 < u_2 < \cdots < u_N$, and let N be so large that the length of each subinterval is less than the number δ in condition (3). Then write

$$\int_a^b G(u) \sin ru\, du = \sum_{n=1}^{N} \int_{u_{n-1}}^{u_n} G(u) \sin ru\, du$$

$$= \sum_{n=1}^{N} \int_{u_{n-1}}^{u_n} [G(u) - G(u_n)] \sin ru\, du + \sum_{n=1}^{N} G(u_n) \int_{u_{n-1}}^{u_n} \sin ru\, du,$$

or

$$\left| \int_a^b G(u) \sin ru\, du \right| \tag{4}$$

$$\le \sum_{n=1}^{N} \int_{u_{n-1}}^{u_n} |G(u) - G(u_n)|\, |\sin ru|\, du + \sum_{n=1}^{N} |G(u_n)| \left| \int_{u_{n-1}}^{u_n} \sin ru\, du \right|.$$

In view of condition (3) and the fact that $|\sin ru| \le 1$, it is easy to see that

$$\int_{u_{n-1}}^{u_n} |G(u) - G(u_n)|\, |\sin ru|\, du < \frac{\varepsilon_0}{2(b-a)} \frac{b-a}{N} = \frac{\varepsilon_0}{2N} \qquad (n = 1, 2, \ldots, N).$$

Also, since $G(u)$ is continuous on the closed interval $a \le u \le b$, it is bounded there; that is, there is a positive number M such that $|G(u)| \le M$ for all u between a and b inclusive. Furthermore,

$$\left| \int_{u_{n-1}}^{u_n} \sin ru\, du \right| \le \frac{|\cos ru_n| + |\cos ru_{n-1}|}{r} \le \frac{2}{r} \qquad (n = 1, 2, \ldots, N),$$

where it is understood that $r > 0$. With these observations, we find that inequality (4) yields the statement

$$\left| \int_a^b G(u) \sin ru\, du \right| < \frac{\varepsilon_0}{2} + \frac{2MN}{r}.$$

we note that, in particular,

$$\lim_{m\to\infty}\int_0^{(m+1/2)\pi}\frac{\sin x}{x}\,dx = L \tag{5}$$

as m passes through positive integers. That is,

$$\lim_{m\to\infty}\int_0^{\pi}\frac{\sin\,[(m+\frac{1}{2})u]}{u}\,du = L, \tag{6}$$

where the substitution $x = (m+\frac{1}{2})u$ has been made for the variable of integration. Observe that equation (6) can be written

$$\lim_{m\to\infty}\int_0^{\pi} F(u)D_m(u)\,du = L, \tag{7}$$

where

$$F(u) = \frac{2\sin(u/2)}{u} \tag{8}$$

and $D_m(u)$ is the Dirichlet kernel (Sec. 40)

$$D_m(u) = \frac{\sin\,[(m+\frac{1}{2})u]}{2\sin(u/2)}.$$

The function $F(u)$, moreover, satisfies the conditions in Lemma 2, Sec. 40, and $F(0+) = 1$ (see Problem 1, Sec. 63). So, by that lemma, limit (7) has the value $\pi/2$; and, by uniqueness of limits, $L = \pi/2$. The proof of Lemma 1 in this section is now complete.

Our second lemma makes direct use of the first one.

Lemma 2. *Suppose that a function $F(u)$ is piecewise continuous on every bounded interval of the positive u axis and that the right-hand derivative $F'_R(0)$ exists. If the improper integral*

$$\int_0^\infty |F(u)|\,du \tag{9}$$

converges, then

$$\lim_{r\to\infty}\int_0^\infty F(u)\frac{\sin ru}{u}\,du = \frac{\pi}{2}F(0+). \tag{10}$$

Observe that the integrand appearing in equation (10) is piecewise continuous on the same intervals as $F(u)$ and that when $u \ge 1$,

$$\left|F(u)\frac{\sin ru}{u}\right| \le |F(u)|$$

Thus the convergence of integral (9) ensures the existence of the integral in equation (10).

We begin the proof of the lemma by demonstrating its validity when the range of integration is replaced by any bounded interval $(0,c)$. That is, we first show that if a function $F(u)$ is piecewise continuous on a bounded interval $(0,c)$ and $F'_R(0)$ exists, then

$$\lim_{r\to\infty}\int_0^c F(u)\frac{\sin ru}{u}\,du = \frac{\pi}{2}F(0+). \tag{11}$$

To prove this, we write

$$\int_0^c F(u)\frac{\sin ru}{u}\,du = I(r) + J(r)$$

where

$$I(r) = \int_0^c \left[\frac{F(u)-F(0+)}{u}\right]\sin ru\,du \quad \to 0$$

and

$$J(r) = \int_0^c F(0+)\frac{\sin ru}{u}\,du.$$

Since the function $G(u) = [F(u) - F(0+)]/u$ is piecewise continuous on the interval $(0,c)$, where $G(0+) = F'_R(0)$, we need only refer to Lemma 1 in Sec. 40 to see that

$$\lim_{r\to\infty} I(r) = 0. \tag{12}$$

On the other hand, if we substitute $x = ru$ into the integral representing $J(r)$ and apply Lemma 1 of the present section, we find that

$$\lim_{r\to\infty} J(r) = F(0+)\lim_{r\to\infty}\int_0^{cr}\frac{\sin x}{x}\,dx = F(0+)\frac{\pi}{2}. \tag{13}$$

Limit (11) is evidently now a consequence of limits (12) and (13).

To actually obtain limit (10), we note that

$$\left|\int_c^\infty F(u)\frac{\sin ru}{u}\,du\right| \le \int_c^\infty |F(u)|\,du,$$

where we assume that $c \ge 1$. We then write

$$\left|\int_0^\infty F(u)\frac{\sin ru}{u}\,du - \frac{\pi}{2}F(0+)\right| \tag{14}$$

$$\le \left|\int_0^c F(u)\frac{\sin ru}{u}\,du - \frac{\pi}{2}F(0+)\right| + \int_c^\infty |F(u)|\,du,$$