Parseval's theorem

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Given

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt, \;\; G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(u) g(t-u) du$$

Prove that

$$\mathcal{F}\{f(t) * g(t)\} = F(\omega) G(\omega)$$

Proof:

$$\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \int_{-\infty}^{\infty} g(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) g(\tau) e^{-i\omega(t+\tau)} dt d\tau$$

By changing the variables, we have $(t, \tau) \rightarrow (t, u)$,

$$t + \tau = u, \;\; t = t$$

Therefore,

$$du dt = J(u, t; \tau, t) d\tau dt$$

where Jacobian $J = 1$. We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) g(u-t) e^{-i\omega u} dt du = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(t) g(u-t) dt\} e^{-i\omega u} du$$

That is

$$\int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(t) g(u-t) dt\} e^{-i\omega u} du = \int_{-\infty}^{\infty} \{f(u) * g(u)\} e^{-i\omega u} du$$

Therefore,

$$F(\omega) G(\omega) = \mathcal{F}\{f * g\}$$

By choosing special case, we have

$$f(t) = f(t)$$

$$g(t) = f(-t)$$

$$f * g = \int_{-\infty}^{\infty} f(u) g(t-u) du = \int_{-\infty}^{\infty} f(u) f(-t+u) du$$

$$F(\omega) F(-\omega) = \mid F(\omega) \mid^2$$

$$\frac{1}{2\pi}\int_{\infty}^{\infty} \mid F(\omega) \mid^2 e^{i\omega\tau} d\omega = \int_{-\infty}^{\infty} f(u) f(-\tau+u) du$$

By choosing special case, $\tau = 0$, we have

$$\frac{1}{2\pi}\int^{\infty} \mid F(\omega) \mid^2 d\omega = \int^{\infty} f^2(u) du$$