From Fourier series to Fourier integral

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試由複數 Fourier 展開式導出複數型 Fourier 積分。

解　設 $f(x)$ 為週期 T 之週期函數，則其複數型 Fourier 級數為

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x},\quad \text{其中 } c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\,e^{-i\omega_n\tau}\,d\tau,\quad \omega_n=\frac{2n\pi}{T}$$

將 c_n 併入 $f(x)$ 內，得 $f(x)=\sum_{n=-\infty}^{\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\,e^{-i\omega_n(\tau-x)}\,d\tau$ . . . . ①

而由 $\omega_n=\dfrac{2n\pi}{T}$ 可得

$$\Delta\omega=\omega_{n+1}-\omega_n=\frac{2(n+1)\pi}{T}-\frac{2n\pi}{T}=\frac{2\pi}{T}\ \Rightarrow\ \frac{1}{T}=\frac{\Delta\omega}{2\pi}$$

當 $T\to\infty$ 時, $\Delta\omega\to d\omega$, $\omega_n\to\omega$, 代入①式，得

$$\begin{aligned}
f(x)&=\lim_{p\to\infty}\sum_{n=-\infty}^{\infty}\frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)e^{-i\omega_n(\tau-x)}\,d\tau\\
&=\int_{-\infty}^{\infty}\Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega(\tau-x)}\,d\tau\Big]\,d\omega\quad \text{Riemann sum to integral}\\
&=\int_{-\infty}^{\infty}\Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\,d\tau\Big]\,e^{i\omega x}\,d\omega\\
&=\int_{-\infty}^{\infty} F(\omega)\,e^{i\omega x}\,d\omega
\end{aligned}$$

其中 $F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\,d\tau=\dfrac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$

Fourier transform:

$$F(\omega)=\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\,d\tau=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

Inverse Fourier transform:

$$F_i(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{i\omega\tau}\,d\tau=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$$