Problems (Section 2–14)

1. Determine the set of values of x for which the sequence converges and find the limit for these values:

a) $\{e^{-nx}\}$ b) $\{\ln nx\}$ c) $\left\{\frac{1}{x^n}\right\}$ d) $\left\{\frac{x^n+1}{x^n+2}\right\}$

e) $\left\{1+\frac{x}{2}+\cdots+\left(\frac{x}{2}\right)^n\right\}$ f) $\{1+e^{-x}+e^{-2x}+\cdots+e^{-nx}\}$

2. Find the set of x for which the series converges and state the radius of convergence:

a) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n x^n$ b) $\sum_{n=0}^{\infty}\left(\frac{x}{e}\right)^n$ c) $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$

d) $\sum_{n=2}^{\infty}\frac{x^n}{n\ln n}$ e) $\sum_{n=0}^{\infty}\frac{(n+1)x^n}{n!}$ f) $\sum_{n=0}^{\infty}\frac{1+(-1)^n}{n!}x^n$

g) $\sum_{n=0}^{\infty}\frac{2n-1}{n^2+1}\frac{x^n}{2^n}$ h) $\sum_{n=2}^{\infty}\frac{n+1}{n\ln n}x^n$ i) $\sum_{n=0}^{\infty} n!\,x^n$

j) $\sum_{n=1}^{\infty}\frac{x^n}{n^n}$ k) $\sum_{n=0}^{\infty}\frac{(x-3)^n}{n^2+1}$ l) $\sum_{n=0}^{\infty}\frac{(x+1)^n}{5^n}$

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m) $\sum_{n=1}^{\infty}\frac{2\cdot 4\cdots(2n)}{4\cdot 7\cdots(3n+1)}(x-2)^n$ n) $\sum_{n=0}^{\infty}\frac{n!+1}{(n!+2)(n!+3)}(x-1)^n$

3. Find the set of x for which the series converges:

a) $\sum_{n=0}^{\infty}\frac{1}{x^n}$ b) $\sum_{n=0}^{\infty}\frac{3^n}{x^n}$ c) $\sum_{n=0}^{\infty}e^{-nx}$

d) $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ test e) $\sum_{n=1}^{\infty}\left(\frac{x+1}{x+2}\right)^n$ f) $\sum_{n=0}^{\infty}n\left(\frac{x}{x-1}\right)^n$

5. Let the power series $\sum_{n=0}^{\infty}c_n x^n$ be given. Show that if the limit exists, then

a) $r^* = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ b) $r^* = \lim_{n\to\infty}\frac{1}{|c_n|^{1/n}}$

ratio test

2–15 UNIFORM CONVERGENCE

If a sequence or series of functions converges over an interval, it may converge much more rapidly at some points than at others. We here explore this question for a series $\sum_{n=1}^{\infty}u_n(x)$ with sum $f(x)$ and corresponding partial sums $S_n(x)$. The discussion for

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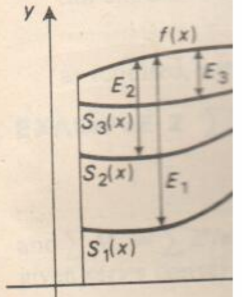
Uniform convergence



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an arbitrary sequence of functions $\{f_n(x)\}$ is the same as that for the sequence $\{S_n(x)\}$.

We introduce the idea of uniform convergence by considering two figures that illustrate two ways in which $S_n(x)$ can approach $f(x)$ (see Fig. 2-9). In each case we are especially interested in the maximum error E_n in the approximation of $f(x)$ by $S_n(x)$; this is shown for $n = 1, 2, 3$ for the two cases. In case (a), $S_2(x)$ is much closer to $f(x)$ than $S_1(x)$ for *all x in the interval*, and $S_3(x)$ is even closer throughout. For case (b), $S_2(x)$ is closer to $f(x)$ than $S_1(x)$ for some (but not all) x in the interval, and $S_3(x)$ is closer still for some (but not all) x. In case (a), the maximum errors E_1, E_2, E_3 are decreasing, and with the process continuing as suggested, approach 0 as $n \to \infty$. In case (b), the maximum errors E_1, E_2, E_3 are all about the same size, however we can see that as n increases the sharp dips in the graphs move over to the left, becoming narrower and narrower, so that $S_n(x)$ does have $f(x)$ as limit for each fixed x. Uniform convergence is illustrated by case (a), nonuniform convergence by case (b).

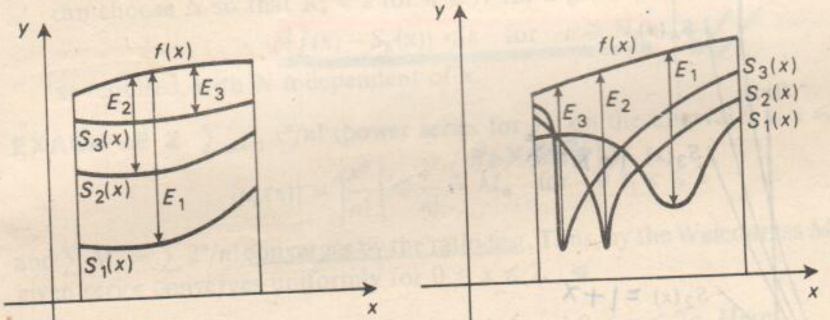

Fig. 2-9. (a) Uniform convergence; (b) nonuniform convergence.

We can define *uniform convergence over an interval $a \leq x \leq b$* as convergence such that the *maximum error over the interval approaches* 0 *as* $n \to \infty$; that is,

$$\max_{a \leq x \leq b} |f(x) - S_n(x)| \to 0 \quad \text{as} \quad n \to \infty. \tag{2-150}$$

This definition is adequate for most applications, namely those in which $f(x)$ and $S_n(x)$ are continuous for all n on the closed interval $a \leq x \leq b$. For more general situations, the definition becomes as follows: for every $\varepsilon > 0$, there is an N, independent of x, such that $|f(x) - S_n(x)| < \varepsilon$ for $n \geq N$ for all x on the given interval. It can be shown that the sum of a uniformly convergent series of continuous functions is continuous. Thus for a convergent series $\sum u_n(x)$ of continuous functions on a closed interval $a \leq x \leq b$ the two definitions of uniform convergence agree.

Uniform convergence is very important for computer calculations. When a series converges uniformly over a given interval, the sum can be computed to desired