1. Find the Laplace transform of $\sqrt{t}$ and $\frac{1}{\sqrt{t}}$,

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

(a). Method 1:
Set $\mathcal{L}\{\sqrt{t}\} = P(s), \mathcal{L}\{\frac{1}{\sqrt{t}}\} = Q(s)$,

$$\mathcal{L}\{p(t)\} = P(s) \Rightarrow \mathcal{L}\{p'(t)\} = s[P(s)] - p(0), p(t) = \sqrt{t}$$

$$\mathcal{L}\{q(t)\} = Q(s) \Rightarrow \mathcal{L}\{tq(t)\} = -(Q'(s)), q(t) = \frac{1}{\sqrt{t}}$$

we have

$$\mathcal{L}\{\sqrt{t}\} = P(s) \Rightarrow \mathcal{L}\{\frac{1}{2\sqrt{t}}\} = s[P(s)] - \sqrt{0} = \frac{1}{2}Q(s)$$

$$\mathcal{L}\{\frac{1}{\sqrt{t}}\} = Q(s) \Rightarrow \mathcal{L}\{t\frac{1}{\sqrt{t}} = \sqrt{t}\} = -(Q'(s)) = P(s)$$

$$\Rightarrow s[-Q'(s)] = \frac{1}{2}Q(s)$$

By solving the 1st order ODE for $Q(s)$, we have

$$\Rightarrow Q(s) = \frac{k}{\sqrt{s}}, P(s) = \frac{k}{2s^{3/2}}, \quad k \in \text{constant}$$

Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt$$

if $s = 1$

$$\mathcal{L}\{t^{\frac{1}{2}}\} = \int_0^\infty t^{-\frac{1}{2}}e^{-st}dt = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt = \Gamma(\frac{1}{2})$$

set $t = v^2$, $dt = 2v \ \ dv$

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-v^2}dv$$

if

$$\int_0^\infty e^{-x^2}dx = C, \int_0^\infty e^{-y^2}dy = C$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = C^2$$

and $x = r cos\theta, y = r sin\theta$

$$\Rightarrow \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = -\frac{1}{2} e^{r^2}|_0^\infty = \frac{\pi}{4} = C^2$$

$$\Rightarrow \int_0^\infty e^{-x^2} dx = C = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}, k = \sqrt{\pi}$$

$$\Rightarrow Q(s) = \sqrt{\frac{\pi}{s}}, P(s) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

(b). Method 2:

$$Hint : \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

Set $x^2 = ts$

$$\Rightarrow 2x\ \ dx = s\ \ dt \Rightarrow dx = \frac{s}{2x} dt$$

$$\Rightarrow \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-st} \frac{s}{2x} dt$$

$$= \int_0^\infty e^{-st} \frac{s}{2\sqrt{ts}} dt = \frac{\sqrt{s}}{2} \int_0^\infty \frac{1}{\sqrt{t}} e^{-st} dt = \frac{1}{2}\sqrt{\pi}$$

$$\Rightarrow \int_0^\infty \frac{1}{\sqrt{t}} e^{-st} dt = \sqrt{\frac{\pi}{s}}$$

$$\Rightarrow \int_0^\infty \sqrt{t} e^{-st} dt = \frac{\sqrt{\pi}}{2s^{3/2}}$$