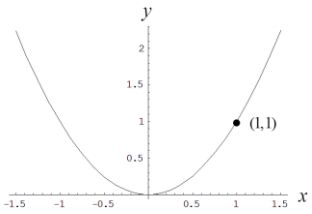
工數第一次大考(Vector calculus) 19:00-21:00 Apr. 17, 2014

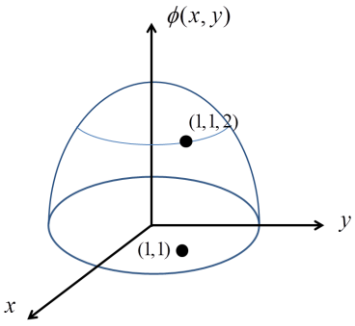
日期：2014 年 4 月 17 日　姓名：＿＿＿＿＿＿　學號：＿＿＿＿＿＿

1. 方向導微 **(11%)**



(1) $y=x^2$ 如圖所示，請求出在 $(1,1)$ 處的切線斜率及切線方程式。

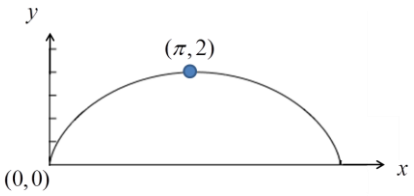
(2) 如右圖，求在 $(x,y,\phi(x,y))=(1,1,2)$ 處的切平面方程式 $\phi=f(x,y)=4-x^2-y^2$ ？



請求出 $\phi(1,1)$ 之方向導微 $\dfrac{\partial\phi}{\partial n}=\nabla\phi\cdot\underset{\sim}{n}=?$ 其中 $\underset{\sim}{n}=(\cos 45^\circ,\sin 45^\circ)$.

2. 擺線曲率計算 (請至少使用兩種方法求解) **(10%)**

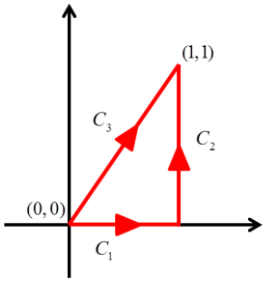
若一曲線(擺線) $\begin{cases}x(t)=t-\sin t\\ y(t)=1-\cos t\end{cases}$



請算出在 $(\pi,2)$ 與之曲率及曲率半徑。

3. Green 定理 與 保守力場 **(15%)**

(a) 如右圖(注意方向)，分別算出 $\int_{C_1}x\,dx+y\,dy=?$ 與 $\int_{C_2}x\,dx+y\,dy=?$ 與

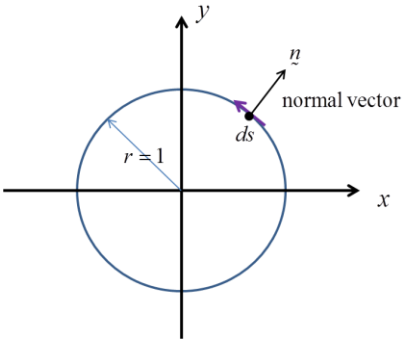


$\int_{C_3}x\,dx+y\,dy=?$ (開放線段), 並使用 Green 定理計算

$\oint_{C_1+C_2-C_3}x\,dx+y\,dy=?$ (封閉曲線)

(b) 此力場 (x,y) 是否為保守力場?

4. Gauss(散度)定理 (面積($2D$) $\leftrightarrow$ 曲線($1D$)) **(18%)**

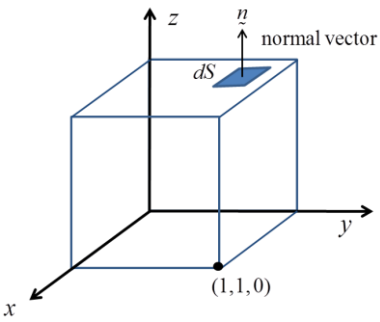


請算出 $\oint\underset{\sim}{v}\cdot\underset{\sim}{n}\,ds=?$

where $\underset{\sim}{n}$ is the unit normal vector of ds and $\underset{\sim}{v}$ is the velocity vector.

(1) $\underset{\sim}{v}=(x,y)$　(2) $\underset{\sim}{v}=(x^2,y^2)$

Gauss(散度)定理 (體積($3D$) $\leftrightarrow$ 曲面($2D$))



請算出 $\iint\underset{\sim}{v}\cdot\underset{\sim}{n}\,dS=?$

where $\underset{\sim}{n}$ is the unit normal vector of dS and $\underset{\sim}{v}$ is the velocity vector.

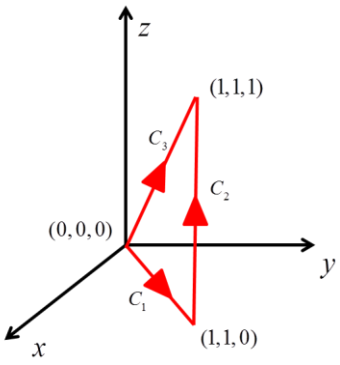
(1) $\underset{\sim}{v}=(x,y,z)$　(2) $\underset{\sim}{v}=(x^2,y^2,z^2)$

5. Stokes 定理 與 保守力場 **(15%)**

(a) 如右圖(注意方向)，分別算出 $\int_{C_1} x\,dx+y\,dy+z\,dz=?$ 與

$\int_{C_2} x\,dx+y\,dy+z\,dz=?$ 與 $\int_{C_3} x\,dx+y\,dy+z\,dz=?$ (開放線段),

並使用 Stoke 定理計算 $\oint_{C_1+C_2-C_3} x\,dx+y\,dy+z\,dz=?$ (封閉曲線)



(b) 此力場 (x,y,z) 是否為保守力場?

6. (a) 3D Frenet formula (正算) **(35%)**

Given a spatial curve described by time-like parameter as follows：

$$\begin{cases} x(t)=\cos(\dfrac{\sqrt{2}}{2}t) \\ y(t)=\sin(\dfrac{\sqrt{2}}{2}t) \\ z(t)=\dfrac{\sqrt{2}}{2}t \end{cases} \Rightarrow \begin{cases} X(s) \\ Y(s) \\ Z(s) \end{cases} ?$$

Try to find the following solutions:

（1） Try to transform the time-like into space-like parameter. ($t \to s$, where s is arc length)

（2） Find the unit tangential vector, normal vector and bi-normal vector of the curve, $\tau(s), \upsilon(s)\ and\ \beta(s)$.

（3） Please determine the radius of curvature for ρ and σ, write the Frenet foumula.

（4） Please determine $(\dot{\tau}\times\dot{\nu})\cdot\dot{\beta}=?$ and $\tau\cdot\dot{\tau}=?$

(b) 2D Frenet formula (反算)

Given $\rho=\sqrt{3}$ and $\sigma=\infty$,

Frenet formula reduces to $\begin{Bmatrix} \dot{\underset{\sim}{\tau}} \\ \dot{\underset{\sim}{\nu}} \end{Bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{3}} \\ \dfrac{-1}{\sqrt{3}} & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\tau} \\ \underset{\sim}{\nu} \end{Bmatrix}$, when $\begin{cases} \underset{\sim}{\tau}(0)=(-1,0) \\ \underset{\sim}{\nu}(0)=(0,-1) \end{cases}$.

Try to find the following solutions:

(1) Try to find $\begin{cases} \underset{\sim}{\tau}(s)=? \\ \underset{\sim}{\nu}(s)=? \end{cases}$

(2) Please determine $e^{\mathbf{A}s}$, where $\mathbf{A}=\begin{bmatrix} 0 & \dfrac{1}{\sqrt{3}} \\ \dfrac{-1}{\sqrt{3}} & 0 \end{bmatrix}$

(3) Please determine the space-like parameter representation for the 2-D curve $\begin{cases} x(s)=? \\ y(s)=? \end{cases}$,

when $\begin{cases} x(0)=0 \\ y(0)=\sqrt{3} \end{cases}$.