A Fourier series solution for the transverse vibration response of a beam with a viscous boundary

Vojin Jovanovic
Systems, Integration & Implementation, Smith Bits, A Schlumberger Co., 16740 Hardy Street, Houston, TX 77032, USA

ABSTRACT
This paper presents the generalized Fourier series solution for the transverse vibration of a beam subjected to a viscous boundary. The model of the system produces a non-self-adjoint eigenvalue-like problem which does not yield orthogonal eigenfunctions; therefore, such functions cannot be used to calculate the coefficients of expansion in the Fourier series. Furthermore, the eigenfunctions and eigenvalues are complex valued. Nevertheless, the eigenfunctions can be utilized if the space of the operator is extended and a suitable inner product is defined. The methodology presented in this paper utilizes Hilbert space methods and is applicable in general to other problems of this type. As an adjunct to the theoretical discussion, the results from numerical simulations are presented.

1. Introduction

This paper presents an investigation of the problem of the transverse vibration of a beam with a viscous boundary. This type of problem arises in situations when viscous dampers are used to control the displacement structures. Unfortunately, the dampers introduce boundary conditions which drastically change the mathematical nature of the problem. As a result, non-self-adjointness arises which renders the problem unsolvable with the usual method of separation of variables. This in turn imposes a need for more general procedures.

Attempts to treat non-self-adjoint boundary conditions in the engineering literature are sparse, perhaps because the vast majority of cases lead to self-adjoint mathematical descriptions. However, a few references exist such (e.g., [1–5]) which provide certain insights into the non-self-adjoint nature of longitudinal vibrations of bars and transverse vibrations of beams. Refs. [1–3] treat longitudinal vibrations of a bar. Ref. [3] demonstrates how difficult it is to get a closed form solution even in the simple case of the wave equation with only harmonic input forcing. Ref. [4] discusses the bending of a simply supported beam with torsional dampers at each end and derives appropriate orthogonality conditions to decouple the equations of motion. Ref. [5] closely follows the methodology laid out in [4] in order to apply Galerkin’s method on the distributionally damped clamped-damper beam. However, application of the derived orthogonality conditions in the process of decoupling the equations of motion there appears to be not without problems. In addition, both Refs. [4,5] only discuss the response to a force and disregard the initial conditions of the beam. Here we take a different approach and avoid these deficiencies by utilizing Hilbert space formalism. This involves extending the definition of the linear operator and defining a suitable inner product to achieve orthogonality of eigenfunctions which in turn enables us to calculate the coefficients of a Fourier’s series. Consequently we derive an orthogonality condition reminiscent of the standard modal analysis and as a result achieve a complete decoupling of the system. This enables us to obtain a closed form solution for the problem which includes specified...
initial conditions of the beam. We therefore provide a complete closed form solution to the problem which, to the best of our
knowledge, has not been obtained to date.

We organize the presentation as follows. We first state the problem and provide its mathematical description in the form
of a partial differential equation. From this equation we derive a boundary value problem (BVP) and determine the
eigenfunctions and eigenvalues (which are complex due to non-self-adjointness). We then analyze the differential operator
and recast the BVP into a state-space self-adjoint form which yields orthogonal eigenvectors. We then perform the usual
modal analysis step together with the Laplace transform and write the response of the system in terms of an eigenfunction
expansion. Finally, we provide numerical illustrations.

2. Problem statement

We begin with the problem statement of the system. Fig. 1 depicts a beam fixed on one and suspended with a damper on
the other end. The symbols \( \rho, A_0, E \) and \( I \) represent the density of the beam, constant cross-sectional area, the modulus of
elasticity and the moment of inertia respectively. The coefficient \( c_1 \) represents the damping coefficient of the damper.

The equation of motion is provided as
\[
c_2^2 \frac{\partial^4 u(x,t)}{\partial x^4} + \frac{\partial^2 u(x,t)}{\partial t^2} = p(x,t), \quad 0 < x < L,
\]
with its associated boundary conditions
\[
\begin{align*}
    u(0,t) &= 0, \\
    \frac{\partial u(0,t)}{\partial x} &= 0, \\
    \frac{\partial^2 u(L,t)}{\partial x^2} &= 0, \\
    \frac{\partial^3 u(L,t)}{\partial x^3} &= h \frac{\partial u(L,t)}{\partial t}.
\end{align*}
\]
and initial conditions
\[
\begin{align*}
    u(x,0) &= f(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = g(x),
\end{align*}
\]
where \( c^2 = EI/\rho A_0 \) and \( h = c_1/El \). \( f(x) \) and \( g(x) \) are assumed real square-integrable functions representing the initial
displacement and velocity of the beam, respectively. \( p(x,t) \) is the force per unit mass distributed along the length of the beam,
while \( u(x,t) \) is a transverse displacement coordinate as shown in Fig. 1. We note that we must have \( f(0)=0 \) and \( g(0)=0 \).

3. Eigenvalues and eigenfunctions

The equation of motion can be solved using the separation of variables method by assuming
\[
u(x,t) = \varphi(x)e^{st},
\]
Substituting this into Eq. (1) with no forcing function on the right-hand side yields the following boundary value problem (BVP):
\[
\begin{align*}
    \varphi''(x) + \frac{j_2}{c_2^2} \varphi(x) &= 0, \\
    \varphi(0) &= 0, \\
    \varphi''(0) &= 0, \\
    \varphi''(L) &= h \varphi(L).
\end{align*}
\]
To solve this eigenvalue problem we assume
\[
\varphi(x) = Ce^{\lambda x},
\]
where \( C \) is an arbitrary (possibly complex) constant. Substituting this into Eq. (5) leads to
\[
j^4 + \frac{j_2^2}{c_2^2} = 0.
\]
If for brevity we define $\beta^2 = -i\lambda/c$ where $i = \sqrt{-1}$ then the four roots of Eq. (7) can be written as $s_{1,2} = \pm \beta$, and $s_{3,4} = \pm i\beta$. In view of Eq. (6) the nontrivial solution to Eq. (5) is

$$\phi(x) = C_1 e^{i\beta x} + C_2 e^{-i\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

and is usually referred to as an eigenfunction. Substituting this eigenfunction into the boundary conditions of Eq. (5) we obtain a system of four homogeneous equations written in matrix form as

$$\begin{bmatrix}
1 & 1 & 1 & 1 \\
\beta & -\beta & i\beta & -i\beta \\
\beta^2 e^{i\beta L} & -\beta^2 e^{-i\beta L} & -\beta^2 e^{i\beta L} & -\beta^2 e^{-i\beta L} \\
(\beta^3 - h\lambda)e^{i\beta L} & (-\beta^3 - h\lambda)e^{-i\beta L} & (-\beta^3 - h\lambda)e^{i\beta L} & (-i\beta^3 - h\lambda)e^{-i\beta L}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}$$

(9)

For a nontrivial solution to exist, the determinant of the matrix of this system must be equal to zero. Writing this determinant in terms of $\beta$ and recognizing that $\beta \neq 0$ since there is no rigid body mode present in the system yields

$$\beta = \frac{ic}{h} \frac{(\cosh(\beta L) \sinh(\beta L) - \sinh(\beta L) \cos(\beta L))}{1 + \cosh(\beta L) \cos(\beta L)}$$

(10)

The solution for this equation must be obtained numerically. The roots of the equation will be $\pm \beta$ and $\pm i\beta$ where the bar over $\beta$ represents the complex conjugate. In view of our definition for $\beta^2$, the roots $\pm \beta$ will lead to $\lambda$ and $\pm i\beta$ will lead to $\lambda$. Therefore, we can select only one solution from each pair as relevant and discard the other. These two solutions used in Eq. (8) provide an appropriate vector basis for spanning the Hilbert space. The other two lead to another possible vector basis. This means that for each eigenvalue $\lambda$ there will be one and only one associated $\beta$ to use for each eigenfunction multiplying terms $e^{i\beta x}$.

Note that the BVP is non-self-adjoint (to be proved in later sections) which is why the eigenvalues must be complex. We choose the indexing of these eigenvalues as follows: $\lambda_1, \lambda_2, \lambda_3, \ldots$ and their complex conjugates $\lambda_{-1} = \bar{\lambda}_1, \lambda_{-2} = \bar{\lambda}_2, \lambda_{-3} = \bar{\lambda}_3, \ldots$

Solving the system in Eq. (9) for $C_2, C_3$ and $C_4$ in terms of $C_1$ yields

$$C_2 = \frac{C_1(ie^{i\beta L} + ie^{-i\beta L} + e^{i\beta L} + e^{-i\beta L})}{ie^{-i\beta L} + ie^{i\beta L} + e^{-i\beta L} + e^{i\beta L}} = C_1 D_2, \quad C_3 = \frac{C_1(ie^{i\beta L} - e^{-i\beta L} + ie^{i\beta L} - e^{-i\beta L})}{ie^{-i\beta L} + ie^{i\beta L} + e^{-i\beta L} + e^{i\beta L}} = C_1 D_3, \quad C_4 = -\frac{C_1(ie^{-i\beta L} + ie^{i\beta L} + e^{-i\beta L} + e^{i\beta L})}{ie^{-i\beta L} + ie^{i\beta L} + e^{-i\beta L} + e^{i\beta L}} = C_1 D_4,$$

(11)

where $D_2, D_3$ and $D_4$ are constants depending on $\beta$. We substitute these into Eq. (8) and write the eigenfunctions as

$$\phi_r(x) = C_1 (e^{i\beta L}x + D_2 e^{-i\beta L}x + D_3 e^{i\beta L}x + D_4 e^{-i\beta L}x), \quad r = \pm 1, \pm 2, \pm 3, \ldots,$$

keeping in mind that $\beta_r$ refers to $\lambda_r$. The constant $C_1$ may be specified arbitrarily. We note that eigenfunctions are complex-valued rendering BVP non-self-adjoint and making the eigenfunctions non-orthogonal.

We also note that Eq. (10) contains in itself two textbook cases allowing the verification of the model: a clamped–free and clamped–pinned beam. This can be seen by taking the limits of Eq. (10) as $h \to 0$ and $h \to \infty$. In the first case, after rearranging the equation and keeping in mind that $\beta \neq 0$, we have

$$\lim_{h \to 0} \frac{ic}{h} \frac{(\cosh(\beta L) \sinh(\beta L) - \sinh(\beta L) \cos(\beta L))}{1 + \cosh(\beta L) \cos(\beta L)} = 1 + \cosh(\beta L) \cos(\beta L) = 0$$

(13)

which is a characteristic equation for the well known case of a clamped–free beam. In the second case, after rearranging the equation, we have

$$\lim_{h \to \infty} \frac{\beta(1 + \cosh(\beta L) \cos(\beta L))}{ic} = \cosh(\beta L) \sinh(\beta L) - \sinh(\beta L) \cos(\beta L) = 0$$

which leads to the known characteristic equation for a clamped–pinned beam $\tan(\beta L) = \tanh(\beta L)$. Therefore, it can be concluded that the system with a damper at the boundary is “in-between” these two cases with parameter $h$ ranging from zero to infinity. This analysis also reveals that the boundary condition $u(L,t) = 0$ for the clamped–pinned beam is “equivalent” to $\partial^2 u(L,t)/\partial x^2 \to \infty$ for the clamped-damper beam implying that as $h \to \infty$ the clamped-damper beam should behave as the clamped–pinned beam.

4. Adjoint operator and Fourier’s expansion

As was pointed out in the previous section, BVP is not self-adjoint and a direct approach to solving our problem by using eigenfunctions is not possible since they are not orthogonal. Therefore, we proceed further with more general methods developed in the theory of Hilbert spaces. We first recast our equation of motion into a state-space representation of the
first-order form by defining a state and excitation vector
\[ w(x,t) = \begin{pmatrix} \frac{\partial u(x,t)}{\partial t} \\ \frac{\partial^2 u(x,t)}{\partial x^2} \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} p(x,t) \\ 0 \end{pmatrix} \]  
\hspace{1cm} (15)

and a matrix differential operator
\[ T = \begin{bmatrix} 0 & -c^2 \frac{\partial^3}{\partial x^3} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix} \]  
\hspace{1cm} (16)

With these definitions our equation of motion takes the first-order form
\[ \frac{\partial w(x,t)}{\partial t} = Tw(x,t) + q(x,t) \]  
\hspace{1cm} (17)

with initial conditions
\[ w(x,0) = \begin{pmatrix} g(x) \\ f(x) \end{pmatrix} \]  
\hspace{1cm} (18)

and boundary conditions
\[ u(0,t) = 0, \quad \frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial^2 u(L,t)}{\partial x^2} = 0, \quad [-h \ 1] w(L,x) = 0. \]  
\hspace{1cm} (19)

Assuming a separable solution
\[ w(x,t) = u(x)e^{\lambda t}, \]  
\hspace{1cm} (20)

where
\[ u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \]  
\hspace{1cm} (21)

Eq. (17) leads to a BVP of the form
\[ Tu(x) = \lambda u(x) \]  
\hspace{1cm} (21)

where \( \lambda \) is an eigenvalue while \( u(x) \) is an eigenvector associated with the eigenvalue. Comparing this BVP to the one in Eq. (5), we see that the eigenvalue is not contained in the boundary conditions. The problem is still not self-adjoint, but it will be possible now to obtain an orthogonality condition which decouples the state-space representation (17).

We define a vector space
\[ X \equiv \left\{ u(x) : u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, u_1, u_2 : [0,L] \rightarrow \mathbb{C} \land x \in \mathbb{R} \right\}, \]  
\hspace{1cm} (22)

where \( \mathbb{R} \) and \( \mathbb{C} \) stand for the set of real and complex numbers respectively. Therefore, the vector space \( X \) is a 2-tuple of complex valued functions of the real variable \( x \) defined on the interval \([0,L]\). We also define a mapping \( \langle \cdot, \cdot \rangle : X \times X \rightarrow F \) known as the inner product, as
\[ \langle u(x), v(x) \rangle = \int_0^L [\overline{u_1(x)}v_1(x) + \overline{u_2(x)}v_2(x)]dx, \]  
\hspace{1cm} (22)

where the bars over \( u_1 \) and \( u_2 \) represent complex conjugation and \( F \) stands for a scalar field. It can be easily verified that \( \langle u(x), v(x) \rangle \) satisfies positive-definiteness, linearity in the second argument and conjugate-symmetry which are the properties needed for the definition of an inner product. The inner product \( \langle u(x), v(x) \rangle \) together with \( X \) form an inner product space.

We now proceed to determine the adjoint of operator \( T \) denoted by \( T^\ast \) where \( T \) and \( T^\ast \) are defined over some domains \( D(T) \) (or just \( D \)) and \( D^\ast(T) \) (or just \( D^\ast \)), respectively [6]. More specifically, we look for a vector \( g(x) \) such that
\[ \langle g(x), u(x) \rangle = \langle v(x), Tu(x) \rangle \]  
\hspace{1cm} (22)

where \( u \) is any element of \( D \), \( v \) is any element of \( D^\ast \) and
\[ g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}. \]
Applying the definition of the inner product to Eq. (22) yields

\[
\int_0^L \left[ \mathcal{G}_1(x)u_1(x) + \mathcal{G}_2(x)u_2(x) \right] dx = \int_0^L \left[ -\mathcal{P}_1(x)u_2''(x) + \mathcal{P}_2(x)u_1''(x) \right] dx \\
= \left[ -c^2\mathcal{P}_1(x)u_2''(x) + c^2\mathcal{P}_1(x)u_2'(x) - c^2\mathcal{P}_1(x)u_2'(x) + c^2\mathcal{P}_1(x)u_2(x) + \mathcal{P}_2(x)u_1(x) \right]_0^L \\
+ \int_0^L \left[ c^2\mathcal{P}_1(x)u_2(x) - \mathcal{P}_2(x)u_1(x) \right] dx.
\]

Comparing the integral on the left-hand side of Eq. (23) with the integral on the right-hand side, we see that in order for the equation to hold, it must be that \( \mathcal{G}_1(x) = -\mathcal{P}_1(x) \) and \( \mathcal{G}_2(x) = c^2\mathcal{P}_1(x) \). Furthermore, the first term, the so-called "surface term", on the right-hand side of Eq. (23) must be equal to zero. This leads to a definition of \( \mathbf{T}^* \) as

\[
\mathbf{T}^* = \begin{bmatrix}
0 & -\frac{\partial}{\partial x} \\
c^2\frac{\partial^2}{\partial x^2} & 0
\end{bmatrix}
\]  

(24)

We obtain the boundary conditions for the adjoint problem by setting the surface term to zero

\[
\begin{align*}
-c^2\mathcal{P}_1(L)u_2''(L) + c^2\mathcal{P}_1(0)u_2''(0) + c^2\mathcal{P}_1(L)u_2'(0) - c^2\mathcal{P}_1(0)u_2'(0) \\
-c^2\mathcal{P}_1(L)u_2''(L) + c^2\mathcal{P}_1(0)u_2'(0) + \mathcal{P}_2(L)u_1(L) - \mathcal{P}_2(0)u_1(0) = 0
\end{align*}
\]

(25)

Substituting the boundary conditions from Eq. (21) into Eq. (25) and identically equating to zero the terms multiplying \( u_2''(L), u_2'(0), u_2'(0) \) and \( u_1(L) \), we obtain the boundary conditions for the adjoint eigenvalue problem. The adjoint eigenvalue problem then becomes

\[
\mathbf{T}^*\mathbf{v}(x) = \mu\mathbf{v}(x)
\]

\[
v_1(0) = 0, \quad v'_1(0) = 0, \quad v''_1(L) = 0,
\]

\[
v_2(L) = c^2hv_1(L)
\]

(26)

where \( \mu \) is the eigenvalue of operator \( \mathbf{T}^* \). We see immediately that \( \mathbf{T}^* = -\mathbf{T}^T \) implying \( \mathbf{T} \neq \mathbf{T}^* \) and demonstrating further that \( \mathbf{T} \) is a non-self-adjoint operator. This confirms the non-self-adjointness of the BVP, Eq. (5).

When \( \mathbf{u} \) is any element of \( \mathcal{D} \) and \( \mathbf{v} \) is any element of \( \mathcal{D}^* \) where \( \mathcal{D} \) and \( \mathcal{D}^* \) represent the set of eigenvectors satisfying associated boundary conditions for each operator \( \mathbf{T} \) and \( \mathbf{T}^* \) respectively, the following relation is satisfied

\[
\langle \mathbf{v}(x), \mathbf{u}(x) \rangle - \langle \mathbf{T}^*\mathbf{v}(x), \mathbf{u}(x) \rangle = 0
\]

(27)

We now proceed with obtaining the required orthogonality condition. If we substitute Eqs. (21) and (26) into Eq. (27) and use the definition of the inner product, we obtain

\[
(\lambda - \mu) \langle \mathbf{v}(x), \mathbf{u}(x) \rangle = 0
\]

(28)

which implies that either \( \lambda = \mu \) or \( \langle \mathbf{v}(x), \mathbf{u}(x) \rangle = 0 \). Therefore, an eigenvector of \( \mathbf{T} \) corresponding to the eigenvalue \( \lambda \) is orthogonal to every eigenvector of \( \mathbf{T}^* \) for which \( \mu \neq \lambda \).

So far, we have obtained an orthogonality condition which relates \( \lambda \) to \( \mu \) through eigenfunctions, but we still do not know how the set of all \( \lambda \)s and \( \mu \)s are related. To determine that, we proceed as follows. We first state a result which can be proved; i.e., either there exists for any vector \( \mathbf{f}(x) \) a vector \( \mathbf{v}(x) \) in the domain of \( \mathbf{T}^* \) such that \( (\mathbf{T}^* - \lambda I)\mathbf{v}(x) = \mathbf{f}(x) \) where \( I \) is the \( 2 \times 2 \) identity matrix or there exists a nontrivial solution to \( (\mathbf{T}^* - \lambda I)\mathbf{v}(x) = 0 \) and therefore an eigenvector of \( \mathbf{T}^* \) corresponding to value \( \lambda \). Based on the result just stated, we shall show that each eigenvalue \( \lambda \) of operator \( \mathbf{T} \) is also an eigenvalue of \( \mathbf{T}^* \).

Suppose that \( \mathbf{u}(x) \) is an eigenvector of \( \mathbf{T} \) corresponding to the eigenvalue \( \lambda \). Then \( (\mathbf{T} - \lambda I)\mathbf{u}(x) = 0 \). If \( \mathbf{v}(x) \) is any vector in \( \mathcal{D}^* \), then in view of Eq. (27) and the inner product

\[
0 = \langle \mathbf{v}(x), (\mathbf{T} - \lambda I)\mathbf{u}(x) \rangle = \langle \mathbf{T}^* - \lambda I\mathbf{v}(x), \mathbf{u}(x) \rangle.
\]

(29)

This implies that \( \mathbf{u}(x) \) is orthogonal to every vector of the form \( (\mathbf{T} - \lambda I)\mathbf{v}(x) \). If \( (\mathbf{T} - \lambda I)\mathbf{v}(x) \) represented every vector \( \mathbf{f}(x) \) we would have a contradiction, since \( \langle \mathbf{u}(x), \mathbf{f}(x) \rangle = 0 \) for every \( \mathbf{f}(x) \) would imply that \( \mathbf{u}(x) = 0 \). Therefore, the other alternative must hold, that there exists an eigenvector of \( \mathbf{T}^* \) corresponding to \( \lambda \). This means that for every \( \lambda \) of operator \( \mathbf{T} \) there is \( \lambda \) which is an eigenvalue of \( \mathbf{T}^* \). Now, since vector \( \mathbf{u}(x) \) is in the domain of \( \mathbf{T} \), \( \mathbf{T} \mathbf{u}(x) = \lambda \mathbf{u}(x) \) implies that \( \lambda \) is also an eigenvalue of \( \mathbf{T} \). Consequently, \( \lambda = \mathbf{T}^\mathbf{T} \) is also an eigenvalue of \( \mathbf{T}^* \). Thus, we have just shown that the set of \( \lambda \)s of operator \( \mathbf{T} \) is the same as the set of \( \mu \)s of operator \( \mathbf{T}^* \).

To determine the eigenvectors of \( \mathbf{T} \), we revert to the BVP specified in Eq. (5). The eigenvalues of that problem are the same as the eigenvalues of the problem specified in Eq. (21). We see now that for \( \lambda = 0 \) the BVP in Eq. (21) has only a trivial solution which cannot be an eigenvector. This a mathematical explanation of the corresponding non-existent rigid body mode in the
system. Eigenvectors for both operators $T$ and $T^*$ can now be readily obtained in view of Eq. (21) as

$$u_n(x) = \left( \begin{array}{c} u_{1,n}(x) \\ \frac{1}{\lambda_n} u'_{1,n}(x) \end{array} \right),$$

(30)

and in view of Eqs. (26) and (30) as

$$v_n(x) = \left( \begin{array}{c} v_{1,n}(x) \\ \frac{e^2}{\mu_n} v'_{1,n}(x) + \frac{\lambda_n}{\lambda_n^2} u_{1,n}(x) \frac{e^2}{\mu_n} u'_{1,n}(x) \end{array} \right).$$

(31)

The right-hand side of Eq. (31) contains $u_1$ because $v_1$ leads to the same form of the BVP and the eigenfunction obtained in Eq. (12) is the same as $u_1$. We can always normalize these eigenvectors by determining the appropriate constant $C_1$ from Eq. (12) by using the orthogonality condition

$$C_1(\lambda_n) = \frac{1}{\sqrt{\langle v_n, u_n \rangle}} = \frac{1}{\sqrt{\int_0^L 2(D_4 e^{(1-i\beta_n)\xi} + D_3 e^{(1+i\beta_n)\xi} + D_3 D_2 e^{(1-\beta_n)\xi} + D_3 D_2 e^{(1+i\beta_n)\xi} + 2D_2 + 2D_4 D_3) d\xi}}.$$  

(32)

The constant will depend on the eigenvalues however the integral to be evaluated in the denominator of Eq. (32) is derivable in closed form. A normalization constant is preferable because otherwise the scaling of the eigenfunctions becomes disproportionate as experienced in [5].

Vectors $u_n$ and $v_n$ now represent infinite orthonormal bases in $X$ which can be used to expand any two-component vector $F$ into a series

$$F(x) = \sum_{n = -\infty}^{\infty} \alpha_n u_n(x),$$

(33)

where

$$\alpha_n = \langle v_n(x), F(x) \rangle.$$  

(34)

Eq. (33) represents a generalized Fourier series for which it can be shown that the coefficients $\alpha_n$ are the best possible choices in the mean square sense. Furthermore, they are the projections onto the basis $u_n$ and since the projections are never greater than the original, no piece of the Fourier series can be greater than $F$. In the limit as $n \to \pm \infty$, the series in Eq. (33) reconstructs $F$ demonstrating the completeness of the basis.

5. Modal analysis

We proceed by using the results of the previous section and decoupling the system of equations given in Eq. (17). In view of our index selection for eigenvalues from Eq. (12), we assume a separable solution

$$u(x, t) = \sum_{r = -\infty}^{\infty} \sum_{r \neq 0} \eta_r(t) u_r(x).$$

(35)

The substitution of Eq. (35) into the system described by Eq. (17) yields

$$\sum_{r = -\infty}^{\infty} \sum_{r \neq 0} \eta_r(t) u_r(x) = \sum_{r = -\infty}^{\infty} \sum_{r \neq 0} \eta_r(t) u_r(x) + q(x, t).$$

(36)

We now take the inner product of the left-hand and right-hand side of the equation with $v_s(x)$ where $s = \pm 1, \pm 2, \pm 3, \ldots$ to obtain a decoupled set of equations. Reverting back to index $r$ from $s$, we write a decoupled set of equations as

$$\dot{u}_r(t) = \eta_r(t) \lambda_r + \int_0^L p(\xi, t) u_{1,r}(\xi) d\xi, \quad r = \pm 1, \pm 2, \pm 3, \ldots$$

(37)

where $u_{1,r}(x)$ is the first component of $u_r(x)$. Applying the Laplace transform to Eq. (37) we obtain

$$\eta_r(s) = \frac{\eta_r(0)}{s-\lambda_r} + \frac{1}{(s-\lambda_r)^2} \int_0^L \tilde{p}(\xi, s) u_{1,r}(\xi) d\xi, \quad r = \pm 1, \pm 2, \pm 3, \ldots$$

(38)

where $\tilde{\eta}_r$ and $\tilde{p}$ represent Laplace transforms of the time-dependent variables with $s$ being a parameter.
Since, we know that the Laplace transform may be expanded as

$$\mathcal{L}\{w(x,t)\} = \mathcal{L}\left(\frac{\partial u(x,t)}{\partial t}, \frac{\partial u(x,t)}{\partial x}\right) = \sum_{r=-\infty}^{\infty} \frac{e^{-st}}{s} L(\eta_r(t)) u(x)$$

we obtain

$$\left(\frac{s \hat{u}(x,s) - u(x,0)}{s \hat{u}(x,s)}\right) = \sum_{r=-\infty}^{\infty} \frac{\hat{\eta}_r(s)}{s} \left(\frac{u_{1,r}(x)}{u_{2,r}(x)}\right).$$

(40)

The first components on both sides of Eq. (40) provide the Laplace transform of the response of the beam as

$$\hat{u}(x,s) = \frac{u(x,0)}{s} + \sum_{r=-\infty}^{\infty} \frac{\hat{\eta}_r(s)}{s} u_{1,r}(x).$$

(41)

In view of Eq. (38) we obtain

$$\hat{u}(x,s) = \frac{u(x,0)}{s} + \sum_{r=-\infty}^{\infty} \frac{\eta_r(0)}{s} u_{1,r}(x) + \sum_{r=-\infty}^{\infty} \left(\frac{1}{s(S-l_r)} \int_0^l \hat{p}(\xi,s) u_{1,r}(\xi) d\xi\right) u_{1,r}(x).$$

(42)

Taking the inverse Laplace transform of Eq. (42), we find the response in the time domain to be

$$u(x,t) = u(x,0) + \sum_{r=-\infty}^{\infty} \eta_r(0) \left[\frac{1}{l_r} (1 - e^{-t/l_r})\right] u_{1,r}(x) + \sum_{r=-\infty}^{\infty} \left(\int_0^t \left[\frac{1}{l_r} (1 - e^{-t'/l_r})\right] \int_0^l p(\xi,t) u_{1,r}(\xi) d\xi dt\right) u_{1,r}(x),$$

(43)

where \(\eta_r(0)\) can be calculated from the initial conditions

$$w(x,0) = \left(\frac{\partial u(x,0)}{\partial t}, \frac{\partial u(x,0)}{\partial x}\right) = \sum_{r=-\infty}^{\infty} \eta_r(0) \left(\frac{u_{1,r}(x)}{u_{2,r}(x)}\right).$$

(44)

Taking the inner product of \(v_r(x)\) with Eq. (44) and reverting back to index \(r\) from \(s\), we obtain the decoupled coefficients of expansion as

$$\eta_r(0) = \int_0^l \left[ g(\xi) u_{1,r}(\xi) + \frac{C^2}{l_r^2} u_{1,r}(\xi) f(\xi) \right] d\xi, \quad r = \pm 1, \pm 2, \pm 3, \ldots$$

(45)

Simplifying the integral through integration by parts and using the BVP in Eq. (5), we obtain \(\eta_r(0)\) written in terms of \(u_{1,r}\) as

$$\eta_r(0) = \int_0^l \left[ \lambda_r f(\xi) g(\xi) u_{1,r}(\xi) d\xi + C^2 hf(L) u_{1,r}(L) \right], \quad r = \pm 1, \pm 2, \pm 3, \ldots$$

(46)

We now substitute Eq. (46) into Eq. (43) and, in view of the continuity of \(u_{1,r}(\xi)\) and \(p(\xi,t)\) on \([0,L]\) together with interchanging the summation and integral signs, several simplifications are possible (see Appendix A for details):

$$u(x,0) + \sum_{r=-\infty}^{\infty} \int_0^l \lambda_r f(\xi) u_{1,r}(\xi) d\xi \left(-\frac{1}{l_r}\right) u_{1,r}(x) = 0,$$

(47)

$$\sum_{r=-\infty}^{\infty} \left(\int_0^l g(\xi) u_{1,r}(\xi) d\xi\right) u_{1,r}(x) = \int_0^l g(\xi) \sum_{r=-\infty}^{\infty} \left(-\frac{u_{1,r}(\xi) u_{1,r}(X)}{l_r}\right) d\xi = 0,$$

(48)

$$\sum_{r=-\infty}^{\infty} \left(\int_0^l C^2 f(L) h u_{1,r}(L)\right) u_{1,r}(x) = C^2 f(L) h \sum_{r=-\infty}^{\infty} \left(-\frac{u_{1,r}(L) u_{1,r}(X)}{l_r}\right) = 0.$$
\[
\sum_{r=-\infty}^{\infty} \left( \int_{0}^{L} \left( -\frac{1}{L^2} \right) \int_{0}^{L} p(\xi, \tau) u_{1,r}(\xi) d\xi d\tau \right) u_{1,r}(x) = \int_{0}^{L} \int_{0}^{L} p(\xi, \tau) \sum_{r=-\infty}^{\infty} \left( -\frac{u_{1,r}(\xi) u_{1,r}(x)}{L^2} \right) d\xi d\tau = 0. \tag{50}
\]

Finally, in view of Eqs. (47)-(50), we can write the vibratory response as

\[
u(x,t) = \sum_{r=-\infty}^{\infty} \left\{ \int_{0}^{L} u_{1,r}(\xi) \epsilon(\xi) d\xi + c^2 h f(L) u_{1,r}(L) \right\} u_{1,r}(x) e^{ix} + \sum_{r=-\infty}^{\infty} \frac{u_{1,r}(x)}{L^2} \int_{0}^{L} p(\xi, \tau) u_{1,r}(\xi) d\xi d\tau. \tag{51}
\]

It should be pointed out that we are, of course, only interested in the real part of Eq. (51). However, there is no particular need to extract only the real part from the response since the imaginary part must necessarily be zero due to all of the initial conditions and the applied force being real. Therefore, the eigenvectors, even though complex themselves, will expand the real quantities into real numbers.

6. Numerical results

In this section we provide several numerical examples to illustrate the effectiveness of the methodology. We present the mode shapes of the normalized first components of the eigenvectors \( \mathbf{u}(x) \) which are in fact the eigenfunctions of Eq. (12). Fig. 2 depicts the real parts of the first six mode shapes for \( c = 1.5, L = 1.8 \) and \( h = 0.1 \). The values for \( \lambda \) can be found in Table 1.

How well the eigenfunctions approximate any function on \([0, L]\)? Unfortunately, we do not have a closed form solution to compare our series expansion for any future time \( t \). However, it is sufficient to examine the expansion at \( t = 0 \) and compare it to the initial conditions \( f(x) \) and \( g(x) \) since these are the only exact solutions available. To this end we choose \( f(x) = 0.01 x^2 \) and \( g(x) = 0 \) for simplicity.

Using the response from Eq. (51), the displacement error for the beam at time \( t = 0 \) is given as

\[
\varepsilon = f(x) - \left( \sum_{r=-k}^{k} \left\{ \int_{0}^{L} u_{1,r}(\xi) \epsilon(\xi) d\xi + c^2 h f(L) u_{1,r}(L) \right\} u_{1,r}(x) e^{ix} \right). \tag{52}
\]

The error is plotted in Fig. 3 for several values of \( k \). We see from Fig. 3 that the error rapidly reduces as the number of terms in the series is increased. The error is within \( 10^{-5} \) for \( k = 20 \).

Table 1 provides the first twenty eigenvalues with positive imaginary part for different values of the parameter \( h \) with \( c = 1.5 \) and \( L = 1.8 \). We provide this table for verification purposes since it is not straightforward to obtain the eigenvalues by solving Eq. (10). The equation is transcendental with infinitely many solutions close to each other and a numerical procedure like Newton–Raphson can easily miss many with a bad initial guess. The values shown in Table 1 were validated with a finite element implementation in Matlab using a consistent-mass matrix formulation as well as with MSC Nastran. For MSC Nastran simulations, the physical parameters...
Table 1
The first twenty eigenvalues with positive imaginary part for different values of parameter h with \( c=1.5, \ L=1.8 \).

<table>
<thead>
<tr>
<th>Eigenvalues for ( h=0.1 )</th>
<th>Eigenvalues for ( h=0.3 )</th>
<th>Eigenvalues for ( h=0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Re )</td>
<td>( Im )</td>
<td>( Re )</td>
</tr>
<tr>
<td>1</td>
<td>-0.250724873</td>
<td>1.610766288</td>
</tr>
<tr>
<td>2</td>
<td>-0.249747131</td>
<td>10.18801558</td>
</tr>
<tr>
<td>3</td>
<td>-0.249873427</td>
<td>28.55548179</td>
</tr>
<tr>
<td>4</td>
<td>-0.249925588</td>
<td>55.96724838</td>
</tr>
<tr>
<td>5</td>
<td>-0.249951621</td>
<td>92.52295409</td>
</tr>
<tr>
<td>6</td>
<td>-0.249966152</td>
<td>138.2163596</td>
</tr>
<tr>
<td>7</td>
<td>-0.249975032</td>
<td>193.0480652</td>
</tr>
<tr>
<td>8</td>
<td>-0.249980838</td>
<td>257.0181439</td>
</tr>
<tr>
<td>9</td>
<td>-0.249984837</td>
<td>320.1266482</td>
</tr>
<tr>
<td>10</td>
<td>-0.249987706</td>
<td>412.3738082</td>
</tr>
<tr>
<td>11</td>
<td>-0.249989833</td>
<td>503.7590427</td>
</tr>
<tr>
<td>12</td>
<td>-0.249991453</td>
<td>604.282964</td>
</tr>
<tr>
<td>13</td>
<td>-0.249992715</td>
<td>713.9453806</td>
</tr>
<tr>
<td>14</td>
<td>-0.249993717</td>
<td>832.7462986</td>
</tr>
<tr>
<td>15</td>
<td>-0.249994526</td>
<td>960.6857221</td>
</tr>
<tr>
<td>16</td>
<td>-0.249995188</td>
<td>1097.783655</td>
</tr>
<tr>
<td>17</td>
<td>-0.249995737</td>
<td>1243.980098</td>
</tr>
<tr>
<td>18</td>
<td>-0.249996197</td>
<td>1399.335056</td>
</tr>
<tr>
<td>19</td>
<td>-0.249996587</td>
<td>1563.828527</td>
</tr>
<tr>
<td>20</td>
<td>-0.24999692</td>
<td>1737.460515</td>
</tr>
</tbody>
</table>

Fig. 3. Error from Eq. (52) for different values of \( k \) with \( c=1.5, \ L=1.8 \) and \( h=0.1 \).

\[
E = 2 \times 10^{11} \text{N}/\text{m}^2, \quad \rho = 7800 \text{kg/m}^3, \quad I = 8.775 \times 10^{-10} \text{m}^4, \quad A = 10^{-2} \text{m}^2 \quad \text{and the values 17.55, 52.65 and 87.75 N s/m for the damping coefficient } c_1 \text{ were used. The values of } c_1 \text{ correspond to the values } 0.1, 0.3 \text{ and } 0.5 \text{ for the parameter } h.
\]

As expected, Table 1 confirms that greater positive values for the parameter \( h \) lead to decrease of the real part of eigenvalues rendering the beam more stable. This is also depicted in Fig. 4 where we plot the response of the beam and see that the beam loses energy faster as \( h \) increases. Subfigures (a), (b) and (c) represent the response of the system for \( f(x)=0.01 x^2, g(x)=0, p(x,t)=0 \) and \( h=0.1, 0.3 \) and 0.5, respectively. They confirm that the beam displacement vanishes more quickly with increasing values of the parameter \( h \).

To validate Fig. 4, numerical results of analytical expression Eq. (51) with twenty eigenfunctions (for each eigenvalue \( \lambda \) and \( \overline{\lambda} \)) were compared to a Matlab finite element implementation using two hundred elements. The results agree to up to six decimal places which further illustrates the efficiency of the analytical expression.

Please cite this article as: V. Jovanovic, A Fourier series solution for the transverse vibration response of a beam with a viscous boundary, Journal of Sound and Vibration (2010), doi:10.1016/j.jsv.2010.10.007
As a final example we determine the response of the system to a harmonic force $F_0 e^{i\omega t} \delta(x - x_f)$ at some point $x_f$ where $F_0$ is a constant. We use this in Eq. (51) after setting the initial displacement and velocity to zero and obtain

$$ u(x,t) = \sum_{r = -\infty}^{\infty} \frac{e^{i\lambda r}}{\rho A_0} \int_0^t \int_0^t F_0 e^{i\omega r} \delta(x - x_f) u_{1,r}(\zeta) d\tau d\tau + \sum_{r = 0}^{\infty} \frac{e^{i\lambda r} F_0 u_{1,r}(x_f)}{\rho A_0 \lambda r} \left( e^{-i\lambda r - \omega r} - 1 \right) u_{1,r}(x). \quad (53) $$

The real part of the response from Eq. (53) is depicted in Fig. 5 where we conveniently set $F_0 = EI$. The displacement shows the gradual increase of vibrations from the beam at rest towards a steady state vibration with a constant amplitude.

7. Conclusions

In this paper we studied the dynamics of transverse vibrations of a beam subjected to a viscous boundary condition. The system is not self-adjoint which prevents readily obtaining an orthogonality condition. Nevertheless, it was possible to deploy Fourier’s method and derive the complete solution in terms of an eigenfunction expansion. The Fourier coefficients were calculated by extending the space of the operator and recasting the underlying eigenvalue problem into a first-order form.
Transverse vibrations of a beam with a discrete damper attached represents only one type of system characterized by non-self-adjointness. Attaching dampers to any other continuous structure such as bars and plates would introduce a similar situation. To our knowledge such investigations have not been carried out in the engineering literature and only a few attempts to obtain an approximate solution, in the case of the beam, have been recently made. However, the methodology described in this paper should, in general, be applicable with a similar treatment in the case of plates and bars. The case of longitudinal vibrations of a bar with viscous boundaries has already been analyzed by the author and a report on the complete closed form solution is to appear soon.

Appendix A. Summation formulae

A.1

Here we determine several sums that are needed to simplify the vibratory response. We first use the generalized Fourier series to expand $f(x)$ as

$$
\begin{pmatrix}
  f(x) \\
  0
\end{pmatrix}
= \sum_{r \neq 0} \int_{-\infty}^{\infty} f(\zeta) u_{1,r}(\zeta) d\zeta \begin{pmatrix}
  u_{1,r}(x) \\
  u_{2,r}(x)
\end{pmatrix}
$$

(A.1)

to provide us with the Fourier’s coefficients needed in Eq. (47)

$$
f(x) = \sum_{r \neq 0} \int_{-\infty}^{\infty} f(\zeta) u_{1,r}(\zeta) d\zeta u_{1,r}(x).
$$

(A.2)

To further simplify of the general response for Eq. (51), we expand $\begin{pmatrix}
  0 \\
  1
\end{pmatrix}$ in terms of a non-self-adjoint basis

$$
\begin{pmatrix}
  0 \\
  1
\end{pmatrix} = \sum_{r \neq 0} \int_{-\infty}^{\infty} u_{2,r}(\zeta) d\zeta \begin{pmatrix}
  v_{1,r}(\zeta) \\
  v_{2,r}(\zeta)
\end{pmatrix}.
$$

(A.3)

From Eq. (21), we have that $u_{2,r}(x) = u'_{1,r}(x)/\lambda_r$ and $v_{1,r}(x) = u_{1,r}(x)$. In view of the boundary conditions we have

$$
\int_{0}^{L} u_{2,r}(\zeta) d\zeta = \frac{1}{\lambda_r} \int_{0}^{L} u'_{1,r}(x) d\zeta = \frac{1}{\lambda_r} [u_{1,r}(x)]_0^L = \frac{u_{1,r}(L)}{\lambda_r}.
$$

(A.4)

Therefore, the first component of the vector in Eq. (A.3) reduces to

$$
0 = \sum_{r \neq 0} \frac{u_{1,r}(L)}{\lambda_r}.
$$

(A.5)
A.2

We now want to show that the restriction \( u_{1,r}(L) \) in Eq. (A.5) can be relaxed to \( u_{1,r}(\xi) \). Once again, we proceed with the generalized Fourier series by expanding \( \delta(x-\xi) \) as follows:

\[
\begin{align*}
0 &= \sum_{r=-\infty}^{\infty} u_{2,r}(\xi) \begin{pmatrix} u_{1,r}(X) \\ v_{2,r}(X) \end{pmatrix}.
\end{align*}
\] (A.6)

In view of \( u_{2,r}(x) = u'_{1,r}(x)/\lambda_r \) from Eq. (21) and \( v_{1,r}(x) = u_{1,r}(x) \), we now write the first component of Eq. (A.6) as

\[
0 = \sum_{r=-\infty}^{\infty} \frac{u'_{1,r}(\xi)u_{1,r}(x)}{\lambda_r}.
\] (A.7)

Eq. (A.7) is not a usual series since we are expanding a vector containing Dirac’s delta function which is not a function in the ordinary sense but a generalized function. The series does not converge to zero in the classical sense, but oscillates faster and faster as more terms are added. The series is in fact a distribution and therefore it converges distributionally or “weakly” [7]. Nevertheless, the series can be used in the ordinary sense for term-by-term integration. We utilize this convergence in this integrated sense and proceed by integrating Eq. (A.7) on both sides with respect to \( \xi \) to obtain

\[
C(x) = \sum_{r=-\infty}^{\infty} \frac{u_{1,r}(\xi)u_{1,r}(x)}{\lambda_r}
\] (A.8)

where \( C(x) \) is a constant of integration combining the constants of integration from both side of Eq. (A.8). To determine it, we set \( \xi = L \) in Eq. (A.8) and compare this equation to Eq. (A.5) after which we conclude that \( C(x) = 0 \). Finally, we obtain a generalized summation formula

\[
0 = \sum_{r=-\infty}^{\infty} \frac{u_{1,r}(\xi)u_{1,r}(x)}{\lambda_r}
\] (A.9)

which can be used to simplify the response of the system.

References


Please cite this article as: V. Jovanovic, A Fourier series solution for the transverse vibration response of a beam with a viscous boundary, *Journal of Sound and Vibration* (2010), doi:10.1016/j.jsv.2010.10.007