Matrix Decomposition Algorithms Related to the MFS for Axisymmetric Problems

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Abstract In this paper we review some applications of the Method of Fundamental Solutions (MFS) to certain elliptic boundary value problems in rotationally symmetric domains. In particular, we show how efficient matrix decomposition MFS algorithms can be developed for such problems. The efficiency of these algorithms is optimized by using Fast Fourier Transforms.

1 Introduction

The Method of Fundamental Solutions (MFS) is a meshless Trefftz-type (Kita and Kamiya, 1995) boundary method which has become very popular in recent years primarily because of the simplicity with which it can be implemented and, unlike the boundary element method, it does not require an elaborate discretization of the boundary. Also, it can be applied even in the case of domains with irregular boundaries. The advantages and disadvantages of the MFS compared to other numerical methods, implementational details as well as a wide range of applications can be found in the survey papers (Cho et al. 2004; Fairweather 2007; Fairweather and Karageorghis 1998; Fairweather et al. 2003; Golberg and Chen 1999). One interesting application of the MFS is to elliptic boundary value problems in rotationally symmetric domains. In particular, in (Karageorghis and Fairweather 1998, n.d., 2000; Kupradze 1965) the MFS was applied to the solution of axisymmetric acoustics, potential and elasticity and problems, respectively. In these studies, the MFS was used to solve the axisymmetric version of the governing equations which, despite reducing the dimension of the problem by one, led to certain difficulties. In particular, the fundamental solutions of these equations involved the potentially troublesome evaluation of complete elliptic integrals.

Moreover, when the boundary conditions of the problem under consideration were not axisymmetric, this approach required the solution of a sequence of

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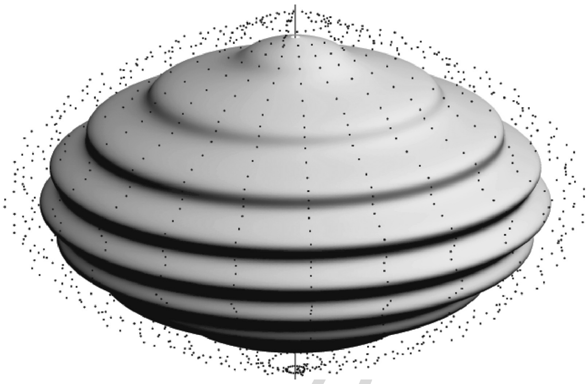

Fig. 1 Typical distribution of singularities on a pseudo–boundary of an axisymmetric domain

problems in order to approximate a finite Fourier sum. Alternatively, the full three-dimensional version of the governing equations may be considered. In such cases, matrix decomposition algorithms (Bialecki and Fairweather 1993) may be developed for the efficient solution of the resulting systems. The solution of three-dimensional harmonic problems in axisymmetric domains was considered in Smyrlis and Karageorghis (2004b) and the corresponding biharmonic case in Fairweather et al. (2005). The solution of such problems in hollow axisymmetric domains is described in Tsangaris et al. (2006a), while the extension of these algorithms to stationary heat conduction problems is presented in Smyrlis and Karageorghis (2006). Finally, elasticity and thermo-elasticity problems are considered in Karageorghis and Smyrlis (2007). All matrix decomposition algorithms algorithms make use of Fast Fourier Transforms (FFTs) and can be seen as generalizations of the basic ideas used for the solution of the corresponding two-dimensional problems in a disk (Smyrlis and Karageorghis 2001, 2004a; Tsangaris et al. 2004, 2006b).

In this paper, we first describe in detail the matrix decomposition algorithm for the solution of three-dimensional harmonic problems and then describe how it can be modified for the solution of three-dimensional elasticity problems. We also show the simplicity of the approach by including a `MATLAB` code implementing the algorithm for three-dimensional harmonic problems. Some numerical results are also included.

2 Axisymmetric Potential Problems

We consider the three-dimensional boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = f \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where, Δ denotes the Laplace operator and f is a given function. The region $\Omega \subset \mathbb{R}^3$ is axisymmetric, which means that it is formed by rotating a region

$\Omega' \in \mathbb{R}^2$ about the z-axis. The boundaries of Ω and Ω' are denoted by $\partial\Omega$ and $\partial\Omega'$, respectively. The solution u is approximated by

$$u_{MN}(\boldsymbol{c}, \boldsymbol{Q}; P) = \sum_{m=1}^{M}\sum_{n=1}^{N} c_{m,n}\, \mathcal{K}_3(P, Q_{m,n}), \qquad P \in \overline{\Omega},$$

where $\boldsymbol{c} \in \mathbb{R}^{MN}$ and $\boldsymbol{Q}$ is a $3MN$-vector containing the coordinates of the sources $Q_{m,n}$, $m = 1, \ldots, M,\ n = 1, \ldots, N$, which lie outside $\overline{\Omega}$. The function $\mathcal{K}_3(P, Q)$ is a fundamental solution of the Laplace equation in $\mathbb{R}^3$ given by

$$\mathcal{K}_3(P, Q) = \frac{1}{4\pi|P - Q|},$$

with $|P - Q|$ denoting the distance between the points P and Q. The singularities $Q_{m,n}$ are fixed on the boundary $\partial\tilde{\Omega}$ of a solid $\tilde{\Omega}$ surrounding Ω. The solid $\tilde{\Omega}$ is generated by the rotation of the planar domain $\tilde{\Omega}'$ which is similar to Ω'. A set of MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on $\partial\Omega$ in the following way. We first choose N points on the boundary $\partial\Omega'$ of Ω'. These can be described by their polar coordinates (r_{P_j}, z_{P_j}), $\ j = 1, \cdots, N$, where r_{P_j} denotes the vertical distance of the point P_j from the z-axis and z_{P_j} denotes the z- coordinate of the point P_j. The points on $\partial\Omega$ are taken to be

$$x_{P_{i,j}} = r_{P_j}\cos\phi_i, \quad y_{P_{i,j}} = r_{P_j}\sin\phi_i, \quad z_{P_{i,j}} = z_{P_j},$$

where $\phi_i = 2(i-1)\pi/M,\ i = 1, \ldots, M$. Similarly, we choose a set of MN singularities $\{Q_{m,n}\}_{i=m,n=1}^{M,N}$ on $\partial\tilde{\Omega}$ by taking $Q_{m,n} = (x_{Q_{m,n}}, y_{Q_{m,n}}, z_{Q_{m,n}})$, and

$$x_{Q_{i,j}} = r_{Q_j}\cos\theta_i, \quad y_{Q_{i,j}} = r_{Q_j}\sin\theta_i, \quad z_{Q_{i,j}} = z_{Q_j},$$

where $\theta_i = 2(\alpha + i - 1)\pi/M,\ i = 1, \ldots, M$. The angular parameter α $(0 \le \alpha < 1/2)$ indicates that the sources are rotated by an angle $2\pi\alpha/M$ in the angular direction. The coefficients $\boldsymbol{c}$ are determined so that the boundary condition is satisfied at the boundary points

$$u_{MN}(\boldsymbol{c}, \boldsymbol{Q}; P_{i,j}) = f(P_{i,j}), \quad i = 1, \ldots, M,\ j = 1, \ldots, N.$$

By ordering both the collocation points $P_{i,j}$ and $Q_{i,j}$ the sources in the following way:

$$k^{\text{th}} \text{ point} = (j-1)M + i,\ i = 1, \ldots, M,\ j = 1, \ldots, N \tag{2}$$

this yields an $MN \times MN$ linear system of the form

$$G\boldsymbol{c} = \boldsymbol{f}, \tag{3}$$

for the coefficients $\boldsymbol{c}$, where

$$\boldsymbol{f} = (f_{11}, f_{21}, \ldots, f_{M1}, \ldots, f_{1N}, \ldots, f_{MN})^T,$$
$$\boldsymbol{c} = (c_{11}, c_{21}, \ldots, c_{M1}, \ldots, c_{1N}, \ldots, c_{MN})^T,$$

and the elements of the matrix G are given by

$$G_{(j-1)M+i,(n-1)M+m} = \frac{1}{4\pi|P_{i,j} - Q_{m,n}|},$$

$i,\ m = 1, \ldots, M,\ \ j,\ n = 1, \ldots, N$. The matrix G has the following block structure

$$G = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix},$$

where the matrices $A_{j,n},\ \ j, n = 1, \cdots, N$, are $M \times M$ circulant matrices each defined by the row

$$a_m^{j,n} = \left(A_{1,m}\right)_{j,n} = \frac{1}{4\pi|P_{1,j} - Q_{m,n}|}, \quad m = 1, \ldots, M \quad j,\ n = 1, \ldots, N.$$

We also write the vectors $\boldsymbol{c}$ and $\boldsymbol{f}$ as

$$\boldsymbol{c} = \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_N \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_N \end{pmatrix}, \tag{4}$$

where $\boldsymbol{c}_n = (c_{1n}, c_{2n}, \ldots, c_{Mn})^T$ and $\boldsymbol{f}_n = (f_{1n}, f_{2n}, \ldots, f_{Mn})^T,\ \ n = 1, \ldots, N$. System (3) can thus be written as

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_N \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_N \end{pmatrix}.$$

If we define U_M to be the unitary $M \times M$ Fourier matrix which is the conjugate of the matrix

$$U_M^* = \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \cdots & \omega^{(M-1)(M-1)} \end{pmatrix}, \quad \text{where} \quad \omega = \mathrm{e}^{2\pi \mathrm{i}/M},$$

and I_N to be the $N \times N$ identity matrix, pre–multiplication of (3) by $I_N \otimes U_M$ yields

$$(I_N \otimes U_M)\, G \left(I_N \otimes U_M^*\right) (I_N \otimes U_M)\, \boldsymbol{c} = (I_N \otimes U_M)\, \boldsymbol{f},$$

or

$$\tilde{G}\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{f}}, \tag{5}$$

where

$$\tilde{G} = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \vdots & \vdots & & \vdots \\ D_{N1} & D_{N2} & \cdots & D_{NN} \end{pmatrix}, \tag{6}$$

and

$$\tilde{\boldsymbol{c}} = \begin{pmatrix} \tilde{\boldsymbol{c}}_1 \\ \tilde{\boldsymbol{c}}_2 \\ \vdots \\ \tilde{\boldsymbol{c}}_N \end{pmatrix}, \quad \tilde{\boldsymbol{f}} = \begin{pmatrix} \tilde{\boldsymbol{f}}_1 \\ \tilde{\boldsymbol{f}}_2 \\ \vdots \\ \tilde{\boldsymbol{f}}_N \end{pmatrix}. \tag{7}$$

In (6), each of the $M \times M$ matrices $D_{j,n},\ j, n = 1, \cdots, N$ is diagonal which is a result of the properties of circulant matrices and as

$$D_{j,n} = U_M A_{j,n} U_M^*,\ j, n = 1, \cdots, N.$$

Further, if

$$D_{j,n} = \mathrm{diag}\left(d_1^{j,n}, d_2^{j,n}, \ldots, d_M^{j,n}\right),$$

we have, for $j, n = 1, \cdots, N$,

$$d_\ell^{j,n} = \sum_{k=1}^{M} a_k^{j,n} \omega^{(k-1)(\ell-1)}, \ \ \ell = 1, \cdots, M. \tag{8}$$

In (5),

$$\tilde{\boldsymbol{c}} = (I_N \otimes U_M)\,\boldsymbol{c} \quad \text{and} \quad \tilde{\boldsymbol{f}} = (I_N \otimes U_M)\,\boldsymbol{f},$$

and, equivalently, in (7),

$$\tilde{\boldsymbol{c}}_\ell = U_M \boldsymbol{c}_\ell \quad \text{and} \quad \tilde{\boldsymbol{f}}_\ell = U_M \boldsymbol{f}_\ell, \quad \ell = 1, \cdots, N,$$

where the vectors $\boldsymbol{c}_\ell$ and $\boldsymbol{f}_\ell$ are defined in (4). Because of the structure of matrix $\tilde{G}$, the solution of system (5) is equivalent to solving the M systems of order N

$$E_m\,\boldsymbol{x}_m = \boldsymbol{y}_m, \quad m = 1, \cdots, M, \tag{9}$$

where

$$(E_m)_{j,n} = d_m^{j,n}, \ \ j, n = 1, \cdots, N$$

and

$$(\boldsymbol{x}_m)_j = \left(\tilde{\boldsymbol{c}}_j\right)_m, \ (\boldsymbol{y}_m)_j = \left(\tilde{\boldsymbol{f}}_j\right)_m, \ \ j = 1, \cdots, N. \tag{10}$$

The solution of the M systems (9) yields the vectors $\boldsymbol{x}_m, \ m = 1, \cdots, M$ from which we can readily recover the vectors $\tilde{\boldsymbol{c}}_n, \ n = 1, \cdots, N$ from (10). Finally, the vectors $\boldsymbol{c}_n, \ n = 1, \cdots, N$ may be calculated from

$$\boldsymbol{c}_n = U_M^*\,\tilde{\boldsymbol{c}}_n. \tag{11}$$

The algorithm described in the section may thus be summarized as follows:

Step 1. Compute $\tilde{\boldsymbol{f}}_\ell = U_M \boldsymbol{f}_\ell, \quad \ell = 1, \cdots, N$.
Step 2. Construct the diagonal matrices $D_{j,n}$ from (8).
Step 3. Solve the M, $N \times N$ systems (9) to obtain the $\{\boldsymbol{x}_m\}_{m=1}^M$, and subsequently the $\{\tilde{\boldsymbol{c}}_n\}_{n=1}^N$.
Step 4. Recover the vector of coefficients $\boldsymbol{c}$ from (11).

Cost. In Step 1 and Step 4, the operations can be carried out via Fast Fourier Transforms (FFTs) at a cost of order $\mathcal{O}(NM \log M)$ operations. FFTs can also be used for the evaluation of the N^2 matrices $D_{j,n}$ in Step 2 at a cost of $\mathcal{O}(N^2 M \log M)$

operations. The FFTs are performed using the MATLAB commands `fft` and `ifft`. Finally, in Step 3, the cost of solving M complex linear systems of order N via $LU-$factorization with partial pivoting is $\mathcal{O}(M\,N^3)$ operations.

Remark. The algorithm described in this section is different from the one described in Smyrlis and Karageorghis (2004b) the sense that the system (3) is set up differently than the corresponding system in Smyrlis and Karageorghis (2004b). The current ordering leads to a block matrix, where each block is a circulant matrix in contrast to the ordering in Smyrlis and Karageorghis (2004b) which leads to a block circulant matrix. Clearly, the two formulations are equivalent.

3 Axisymmetric Elasticity Problems

We consider the boundary value problem in $\mathbb{R}^3$ governed by the Cauchy–Navier equations of elasticity

$$\begin{cases} (\lambda+\mu)\,u_{k,ki} + \mu\,u_{i,kk} = 0 \text{ in } & \Omega, \\ u_i = f_i & \text{on } \partial\Omega. \end{cases}$$

The region $\Omega \subset \mathbb{R}^3$ is, as in Section 2, axisymmetric. The displacements $\boldsymbol{u} = \big(u_1, u_2, u_3\big)$ at the point $P \in \mathbb{R}^3$ are approximated by

$$u_i^{M,N}(\boldsymbol{c},\,\boldsymbol{Q}\,;\,P) = \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{j=1}^{3} c_{m,n}^{j}\,g_{ij}(P-Q_{m,n}),\ i=1,2,3,$$

where $\boldsymbol{Q} = (Q_{m,n})_{m=1,\ldots,M}^{n=1,\ldots,N}$ with $Q_{m,n} = \big(x_{m,n}^{Q},\,y_{m,n}^{Q},\,z_{m,n}^{Q}\big) \in \mathbb{R}^3$ are the coordinates of the sources. The fundamental solution in this case is a 3×3 matrix defined by

$$g_{ij}(\boldsymbol{x}) = -\frac{3\mu+\lambda}{8\pi\mu\,(2\mu+\lambda)}\cdot\frac{\delta_{ij}}{|\boldsymbol{x}|} - \frac{\mu+\lambda}{8\pi\mu\,(2\mu+\lambda)}\cdot\frac{x_i x_j}{|\boldsymbol{x}|^3}.$$

The discretization of the axisymmetric domain is carried out as in Section 2. The satisfaction of the boundary conditions at the boundary points $\{P_{k,\ell}\}_{k=1,\ell=1}^{M,N}$ yields

$$u_1^{M,N}(\boldsymbol{c},\,\boldsymbol{Q}\,;\,P_{k,\ell}) = f_1(P_{k,\ell}),$$

$$u_2^{M,N}(\boldsymbol{c},\,\boldsymbol{Q}\,;\,P_{k,\ell}) = f_2(P_{k,\ell}),$$

$$u_3^{M,N}(\boldsymbol{c},\,\boldsymbol{Q}\,;\,P_{k,\ell}) = f_3(P_{k,\ell}),$$

$k = 1, \ldots, M,\ \ \ell = 1, \ldots, N$. By ordering both the boundary points and the singularities as in (2), this gives a $3MN \times 3MN$ linear system of the form

$$G\,\boldsymbol{c} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}^1 \\ \boldsymbol{c}^2 \\ \vdots \\ \boldsymbol{c}^N \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}^1 \\ \boldsymbol{f}^2 \\ \vdots \\ \boldsymbol{f}^N \end{pmatrix} = \boldsymbol{f}, \tag{12}$$

where

$$A_{\ell,\nu} = \begin{pmatrix} A^{1,1}_{\ell,\nu} & A^{1,2}_{\ell,\nu} & \cdots & A^{1M,}_{\ell,\nu} \\ A^{2,1}_{\ell,\nu} & A^{2,2}_{\ell,\nu} & \cdots & A^{2,M}_{\ell,\nu} \\ \vdots & \vdots & \ddots & \vdots \\ A^{M,1}_{\ell,\nu} & A^{M,2}_{\ell,\nu} & \cdots & A^{M,M}_{\ell,\nu} \end{pmatrix},$$

where $A^{k\mu}_{\ell,\nu} = g(P_{k,\ell} - Q_{\mu,\nu}) \in \mathbb{R}^{3\times 3}$,

$$\boldsymbol{c}^{\nu} = \left(\boldsymbol{c}^{1,\nu}, \ldots, \boldsymbol{c}^{M,\nu}\right) = \left(c^1_{1,\nu}, c^2_{1,\nu}, c^3_{1,\nu}, c^1_{2,\nu}, c^2_{2,\nu}, c^3_{2,\nu}, \ldots, c^1_{M,\nu}, c^2_{M,\nu}, c^3_{M,\nu}\right),$$

$$\boldsymbol{f}^{\ell} = \left(\boldsymbol{f}^{1,\ell}, \ldots, \boldsymbol{f}^{M,\ell}\right) = \left(f^1_{1,\ell}, f^2_{1,\ell}, f^3_{1,\ell}, f^1_{2,\ell}, f^2_{2,\ell}, f^3_{2,\ell}, \ldots, f^1_{M,\ell}, f^2_{M,\ell}, f^3_{M,\ell}\right),$$

and $f^i_{k,\ell} = f_i(P_{k,\ell})$, $i = 1, 2, 3$, $k, \mu = 1, \ldots, M$ and $\ell, \nu = 1, \ldots, N$. We pre–multiply system (12) by the matrix

$$\mathcal{R} = I_N \otimes R$$

where

$$R = \begin{pmatrix} R_{\vartheta_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & R_{\vartheta_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & R_{\vartheta_{M-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & R_{\vartheta_M} \end{pmatrix},$$

with

$$R_{\vartheta_k} = \begin{pmatrix} \cos\vartheta_k & \sin\vartheta_k & 0 \\ \sin\vartheta_k & -\cos\vartheta_k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\vartheta_k = \mathrm{d}s\,\frac{2\pi(k-1)}{M}$, to obtain, using the properties of $\mathcal{R}$,

$$\mathcal{R}G\boldsymbol{c} = \mathcal{R}G\mathcal{R}\,\mathcal{R}\boldsymbol{c} = \mathcal{R}\boldsymbol{f}, \quad \text{or} \quad \tilde{G}\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{f}}, \tag{13}$$

where $\tilde{G} = \mathcal{R}G\mathcal{R}$, $\tilde{\boldsymbol{c}} = \mathcal{R}\boldsymbol{c}$ and $\tilde{\boldsymbol{f}} = \mathcal{R}\boldsymbol{f}$. The matrix $\tilde{G}$ has the form

$$\tilde{G} = \begin{pmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} & \cdots & \tilde{A}_{1,N} \\ \tilde{A}_{2,1} & \tilde{A}_{2,2} & \cdots & \tilde{A}_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}_{N,1} & \tilde{A}_{N,2} & \cdots & \tilde{A}_{N,N} \end{pmatrix},$$

where

$$\tilde{A}_{\ell,\nu} = RA_{\ell,\nu}R = \begin{pmatrix} \tilde{A}^{1,1}_{\ell,\nu} & \tilde{A}^{1,2}_{\ell,\nu} & \cdots & \tilde{A}^{1,M}_{\ell,\nu} \\ \tilde{A}^{2,1}_{\ell,\nu} & \tilde{A}^{2,2}_{\ell,\nu} & \cdots & \tilde{A}^{2,M}_{\ell,\nu} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}^{M,1}_{\ell,\nu} & \tilde{A}^{M,2}_{\ell,\nu} & \cdots & \tilde{A}^{M,M}_{\ell,\nu} \end{pmatrix}$$

and

$$\tilde{A}^{k,\mu}_{\ell,\nu} = R_{\vartheta_k} A^{k,\mu}_{\ell,\nu} R_{\vartheta_\mu}.$$

At this point we re-order the vectors $\tilde{\boldsymbol{c}}$ and $\tilde{\boldsymbol{f}}$ as $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{f}}$, respectively, where

$$\hat{\boldsymbol{c}} = \begin{pmatrix} \hat{\boldsymbol{c}}^1 \\ \hat{\boldsymbol{c}}^2 \\ \vdots \\ \hat{\boldsymbol{c}}^N \end{pmatrix}, \quad \hat{\boldsymbol{f}} = \begin{pmatrix} \hat{\boldsymbol{f}}^1 \\ \hat{\boldsymbol{f}}^2 \\ \vdots \\ \hat{\boldsymbol{f}}^N \end{pmatrix},$$

where

$$\hat{\boldsymbol{c}}^\nu = \left(\tilde{c}^1_{1,\nu}, \ldots \tilde{c}^1_{M,\nu}, \tilde{c}^2_{1,\nu}, \ldots \tilde{c}^2_{M,\nu}, \tilde{c}^3_{1,\nu}, \ldots \tilde{c}^3_{M,\nu}\right) \in \mathbb{R}^{3M},$$

$$\hat{\boldsymbol{f}}^\nu = \left(\tilde{f}^1_{1,\nu}, \ldots \tilde{f}^1_{M,\nu}, \tilde{f}^2_{1,\nu}, \ldots \tilde{f}^2_{M,\nu}, \tilde{f}^3_{1,\nu}, \ldots \tilde{f}^3_{M,\nu}\right) \in \mathbb{R}^{3M}.$$

With this re-ordering, system (13) becomes

$$\hat{G}\hat{\boldsymbol{c}} = \hat{\boldsymbol{f}}, \tag{14}$$

where

$$\hat{G} = \begin{pmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} & \cdots & \hat{A}_{1,N} \\ \hat{A}_{2,1} & \hat{A}_{2,2} & \cdots & \hat{A}_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{N,1} & \hat{A}_{N,2} & \cdots & \hat{A}_{N,N} \end{pmatrix}$$

where the matrices

$$\hat{A}_{\ell,\nu} = \begin{pmatrix} \hat{A}^{1,1}_{\ell,\nu} & \hat{A}^{1,2}_{\ell,\nu} & \hat{A}^{1,3}_{\ell,\nu} \\ \hat{A}^{2,1}_{\ell,\nu} & \hat{A}^{2,2}_{\ell,\nu} & \hat{A}^{2,3}_{\ell,\nu} \\ \hat{A}^{3,1}_{\ell,\nu} & \hat{A}^{3,2}_{\ell,\nu} & \hat{A}^{3,3}_{\ell,\nu} \end{pmatrix}$$

are $M \times M$, and

$$\left(\hat{A}^{k,\mu}_{\ell,\nu}\right)_{i,j} = \left(\tilde{A}^{i,j}_{\ell,\nu}\right)_{k,\mu}. \tag{15}$$

Each of the $M \times M$ matrices $\hat{A}^{k,\mu}_{\ell,\nu}$ is now circulant (Karageorghis and Smyrlis 2007) and thus premultiplication of (14) by $I_N \otimes I_3 \otimes U_M$ yields

$$(I_N \otimes I_3 \otimes U_M)\,\hat{G}\left(I_N \otimes I_3 \otimes U_M^*\right)(I_N \otimes I_3 \otimes U_M)\,\hat{\boldsymbol{c}} = (I_N \otimes I_3 \otimes U_M)\,\hat{\boldsymbol{f}}, \tag{16}$$

or

$$D\boldsymbol{\gamma} = \boldsymbol{h}, \tag{17}$$

where where

$$D = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \vdots & \vdots & & \vdots \\ D_{N1} & D_{N2} & \cdots & D_{NN} \end{pmatrix},$$

and where the matrices

$$\hat{D}_{\ell,\nu} = \begin{pmatrix} \hat{D}^{1,1}_{\ell,\nu} & \hat{D}^{1,2}_{\ell,\nu} & \hat{D}^{1,3}_{\ell,\nu} \\ \hat{D}^{2,1}_{\ell,\nu} & \hat{D}^{2,2}_{\ell,\nu} & \hat{D}^{2,3}_{\ell,\nu} \\ \hat{D}^{3,1}_{\ell,\nu} & \hat{D}^{3,2}_{\ell,\nu} & \hat{D}^{3,3}_{\ell,\nu}, \end{pmatrix} \tag{18}$$

while $\boldsymbol{\gamma} = (I_N \otimes I_3 \otimes U_M)\,\hat{\boldsymbol{c}}$ and $\boldsymbol{h} = (I_N \otimes I_3 \otimes U_M)\,\hat{\boldsymbol{f}}$. In (18), each of the $M \times M$ matrices $\hat{D}^{k,\mu}_{\ell,\nu} = \operatorname{diag}\left(d^{k,\mu}_{\ell,\nu}{}^{1}, d^{k,\mu}_{\ell,\nu}{}^{2}, \cdots, d^{k,\mu}_{\ell,\nu}{}^{M}\right)$ is diagonal and its elements may be calculated as in (8). Solving (17) is thus equivalent to solving M systems of order $3N$ of the form

$$E_m\,\boldsymbol{x}_m = \boldsymbol{y}_m, \quad m = 1, \cdots, M \tag{19}$$

where for $m = 1, \cdots, M$

$$(E_m)_{i,j} = \left(d^{k,\mu}_{\ell,\nu}{}^{m}\right), \quad i = 3(\ell - 1) + k, \quad j = 3(\nu - 1) + \mu,$$

$\ell, \nu = 1, \cdots, N,\ k, \mu = 1, 2, 3$. The vectors $\boldsymbol{x}_m$ and $\boldsymbol{y}_m$ are defined accordingly. Having obtained $\boldsymbol{\gamma}$ one may recover $\hat{\boldsymbol{c}} = \left(I_N \otimes I_3 \otimes U^*_M\right)\boldsymbol{\gamma}$ and subsequently $\boldsymbol{c}$.

The algorithm described in the section may thus be summarized as follows:

Step 1. Compute $\tilde{\boldsymbol{f}} = \mathcal{R}\boldsymbol{f}$.
Step 2. Construct the first rows of the $M \times M$ submatrices $\hat{A}^{k,\mu}_{\ell,\nu}$, $k, \mu = 1, \ldots, 3$, $\ell, \nu = 1, \ldots, N$ from (15).
Step 3. Compute $\boldsymbol{h} = (I_N \otimes I_3 \otimes U_M)\,\hat{\boldsymbol{f}}$.
Step 4. Construct the matrices $\hat{D}^{k,\mu}_{\ell,\nu}$ using (8).
Step 5. Solve the M systems of order $3N$ in (19)
Step 6. Recover the vector of coefficients $\hat{\boldsymbol{c}} = \left(I_N \otimes I_3 \otimes U^*_M\right)\boldsymbol{\gamma}$ and subsequently $\boldsymbol{c}$.

Cost. In Step 3, Step 4 and Step 6, the operations can be carried out via FFTs at a cost of order $\mathcal{O}(NM \log M)$ operations. In Step 5, the cost of solving M complex linear systems of order $3N$ via $LU-$factorization with partial pivoting is $\mathcal{O}(M\,N^3)$ operations.

Remark. The algorithm described in this section is different from the one described in Karageorghis and Smyrlis (2007) the sense that the system is set up differently than the corresponding system in Karageorghis and Smyrlis (2007). As in Section 2 the two formulations are equivalent.

4 Numerical Results

We considered problem (1) with f corresponding to the exact solution

$$u(x, y, z) = \mathrm{e}^{ax} \cos by \sin cz$$

when Ω is the unit sphere. The absolute value of the maximum error was calculated for various values of $M = N$ on a uniformly distributed set of points (different from the collocation points) on the unit sphere. Plots of the maximum error versus the radius R of the pseudo–boundary are presented in Fig. 2 for $\alpha = 0$ and

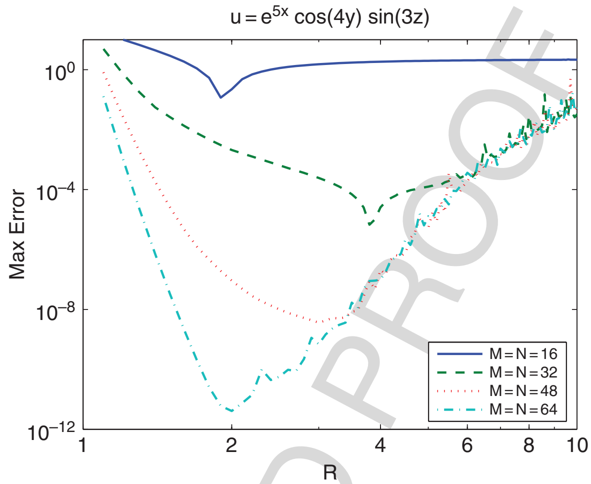

Fig. 2 Maximum error versus R

$a = 5$, $b = 4$, $c = 3$. In all cases we observe that the accuracy of the approximation improves as we increase M. Also, because of ill-conditioning, for larger values of R the accuracy deteriorates. In Fig. 3 we present plots of the maximum error versus the angular parameter α for a fixed pseudo-boundary of radius $R = 1.01$ for $a = 0.5$, $b = 0.4$, $c = 0.3$. These confirm the observations reported in previous studies, i.e. that when the pseudo-boundary is close to the boundary, the variation of the angular parameter does improve the accuracy of the MFS approximation, with a minimum reached for $\alpha \approx 0.25$. In order to show the simplicity of the implementation of the algorithm presented in this work, in the Appendix we present a `MATLAB` code performing the calculations presented.

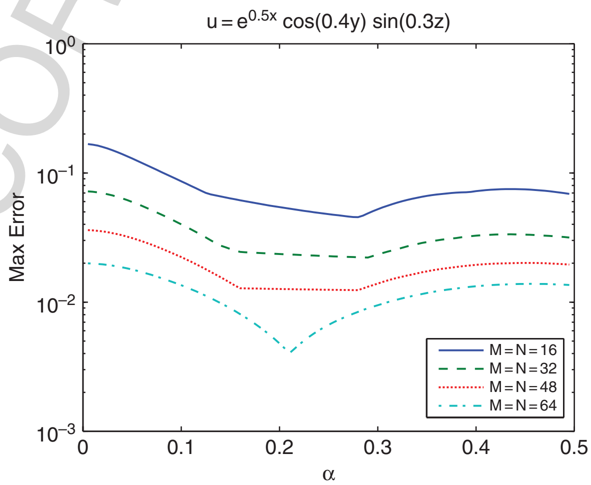

Fig. 3 Maximum error versus α for $R = 1.01$

5 Concluding Remarks

In this paper we describe how the MFS can be efficiently implemented for the solution of problems in rotationally symmetric domains using FFT-based domain decomposition algorithms. We present the algorithm for the cases of the Laplace equation and the Cauchy-Navier equations of elasticity. Numerical results for the former are presented as well as the MATLAB code used for the numerical tests.

Appendix

```
function mfs_mda(f,mp,iter,rp,ds,alfa,n,m)
fi=2*pi/m;th=pi/(n+1);rs=rp+ds;om=ones(1,m);
onm=ones(n,m);omp=ones(mp,mp);at=th*(1:n);
for ii=1:iter
   rs=rp+ii*ds;alfa=.5*ii/(iter+1);af=fi*(0:m-1);
   cat=cos(at);sat=sin(at);ca=cos(af);sa=sin(af);
   ca1=cos(af+fi*alfa);sa1=sin(af+fi*alfa);
   xp=rp*sat'*ca;yp=rp*sat'*sa;zp=rp*cat'*om;
   b=feval(f,xp,yp,zp);xs=rs*sat'*ca1;
   ys=rs*sat'*sa1;zs=rs*cat'*om;
   for in=1:n
      x=xp(in,1)*onm-xs;y=yp(in,1)*onm-ys;
      z=zp(in,1)*onm-zs;r=sqrt(x.^2+y.^2+z.^2);
      aaa(in,:,:)=(1/(4*pi))/r;
   end
   for k=1:n
      fff(k,:)=fft(b(k,:)')/sqrt(m);
   end
   for im=1:m
      for in=1:n
         for jn=1:n
            d(in,jn,:)=ifft(aaa(in,jn,:))*m;
            matr(in,jn)=d(in,jn,im);
         end
      end
      sol=matr\fff(:,im)'; c(:,im)=sol;
   end
   for k=1:n
      ct(k,:)=real(ifft(c(k,:)')*sqrt(m));
   end
   angf=th*(0:mp-1);angt=fi*(1:mp);car=cos(angt);
   sar=sin(angt); caf=cos(angf);saf=sin(angf);
   xpm=rp*sar'*caf;ypm=rp*sar'*saf;
   zpm=rp*car'*ones(1,mp);exa=feval(f,xpm,ypm,zpm);
```

```
    for jn=1:n
       for jm=1:m
          x=xpm-xs(jn,jm)*omp;y=ypm-ys(jn,jm)*omp;
          z=zpm-zs(jn,jm)*omp;r=sqrt(x.^2+y.^2+z.^2);
          approx=approx+((1/(4*pi))*ct(jn,jm)./r);
       end
    end
    error=max(max(approx-exa));
 end
```

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