國立臺灣海洋大學河海工程學系 2000 工程數學 (一) 第二次大考解答

1 (a). Given a spiral curve, we can describe by time-like parameter as follows:

$$x(t) = cos(t), y(t) = sin(t), z(t) = t$$

Please describe the curve by using space-like parameter (arc length s). (5 %)
Ans: $\mathbf{r}(s) = (cos(s/\sqrt{2}), sin(s/\sqrt{2}), s/\sqrt{2})$.
1 (b). Plot the curve from the starting point of $(1, 0, 0)$? (5 %)
Ans: a spring
1 (c). What is the arc length of the curve from $t = 0$ to $t = 2\pi$? (5 %)
Ans: $2\sqrt{2}\pi$.
1 (d). Please determine the radius of curvature for ρ and σ as shown below: (5 %)
Ans: $\rho = 2, \sigma = 2$

$$\begin{Bmatrix} \dot{\hat{\mathbf{t}}} \\ \dot{\hat{\mathbf{n}}} \\ \dot{\hat{\mathbf{b}}} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{\rho} & 0 \\ \frac{-1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & \frac{-1}{\sigma} & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \\ \hat{\mathbf{b}} \end{Bmatrix} \tag{1}$$

1 (e). Determine (5 %)

$$(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}) \cdot \frac{d^3\mathbf{r}}{ds^3} = ?$$

Ans: 1/8

2. Given the radial position vector ($\mathbf{r}$) and radial basis function ($\phi(r)$)

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$r = \sqrt{x^2 + y^2 + z^2},$$

$$\mathbf{b} = \omega\mathbf{k},$$

find (20 %)

$$\nabla \cdot [\phi(r)\mathbf{r}] = 3\phi(r) + r\frac{d\phi(r)}{dr}$$

$$\nabla \times [\phi(r)\mathbf{r}] = \mathbf{0}$$

$$\nabla \cdot \mathbf{r} = 3$$

$$\nabla \times (\mathbf{b} \times \mathbf{r}) = 2\mathbf{b}$$

3. Given a cone with $z = 2\sqrt{x^2 + y^2}$, find the normal vector of the point $(1, 0, 2)$ on the cone. (5 %)
Ans: $\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{k}$.

4. The temperature space field at the point (x, y, z) in space is inversely proportional to the square of the distance from (x, y, z) to the origin $(0, 0, 0)$, i.e., $T(x, y, x) = 1/(x^2 + y^2 + z^2)$. Find the rate of change of T at $(2, 3, 3)$ in the direction of $(3, 1, 1)$. (5 %) In which direction from $(2, 3, 3)$ does the temperature T increase most rapidly ? (5 %) At the point $(2, 3, 3)$ what is the maximum rate of change of T ? (5 %)
Ans: (1) $\frac{-6}{121\sqrt{11}}$, (2). (-2, -3, -3), (3). $\frac{1}{11\sqrt{22}}$.

Table 1: Three coordinate systems (15 %)

1	Curvilinear coordinate	(x, y, z) system	(ρ, ϕ, z) system	(r, θ, ϕ) system
2	$x = x(u_1, u_2, u_3)$	$x = x$	$x = \rho cos(\phi)$	$x = rsin(\theta)cos(\phi)$
3	$y = y(u_1, u_2, u_3)$	$y = y$	$y = \rho sin(\phi)$	$y = rsin(\theta)sin(\phi)$
4	$z = z(u_1, u_2, u_3)$	$z = z$	$z = z$	$z = rcos(\theta)$
5	$h_1 =\mid \frac{\partial \mathbf{r}}{\partial u_1} \mid$	1	1	1
6	$h_2 =\mid \frac{\partial \mathbf{r}}{\partial u_2} \mid$	1	ρ	r
7	$h_3 =\mid \frac{\partial \mathbf{r}}{\partial u_3} \mid$	1	1	$rsin(\theta)$
8	$(ds)^2$	$dx^2 + dy^2 + dz^2$	$d\rho^2 + (\rho d\phi)^2 + dz^2$	$dr^2 + (rd\theta)^2 + r^2 sin^2(\theta)(d\phi)^2$
9	dV	$dxdydz$	$\rho d\rho d\phi dz$	$r^2 sin\theta dr d\theta d\phi$
10	$\nabla\Phi$			

5. For the cylindrical polar coordinates (ρ, ϕ, z), determine (20 %) (1). $\partial\hat{\mathbf{e}}_\rho/\partial\rho = 0$, (2). $\partial\hat{\mathbf{e}}_\phi/\partial\rho = 0$, (3). $\partial\hat{\mathbf{e}}_\rho/\partial\phi = \hat{\mathbf{e}}_\phi$, (4). $\partial\hat{\mathbf{e}}_\phi/\partial\phi = -\hat{\mathbf{e}}_\rho$.
6. Please fill in the following blanks.

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $(ds)^2 = (d\mathbf{r}) \cdot (d\mathbf{r})$, $dV = h_1 h_2 h_3 du_1 du_2 du_3$ and $\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\hat{\mathbf{e}}_3$.