

1 The vector field $\mathbf{a} = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2xz)\mathbf{k}$. Show that \mathbf{a} is conservative field and find the line integral of $\int \mathbf{a} \cdot d\mathbf{r}$ along any line joining (1, 1, 1) and (1, 2, 2). (10 %)

2 Green's theorem : $\int \int_A (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \oint_C (Pdx + Qdy)$

Determine the following line integrals and area integrals as shown in Fig.1 (25 %)

(a). $\frac{1}{2} \oint_C \{-ydx + xdy\}$, (b). $\frac{1}{2} \oint_C |\mathbf{r} \times \hat{\mathbf{t}}| ds$, (c). $\frac{1}{2} \oint_C \mathbf{r} \cdot \hat{\mathbf{n}} ds$, (d). $\int_A (\nabla \times \mathbf{a}) \cdot \mathbf{k} dA$, and (e). $\int_A dA$, where C is the circular boundary of a unit circle, $\hat{\mathbf{t}}$ is the unit tangent vector, $\hat{\mathbf{n}}$ is the unit normal vector, $(ds)^2 = d\mathbf{r} \cdot d\mathbf{r}$, $\mathbf{a} = (-y, x)$ and A is the area of a unit circle.

3 Gauss' theorem: $\int \int_V \nabla \cdot \mathbf{a} dV = \int \int_S \mathbf{a} \cdot d\mathbf{S}$

(a). Find the volume enclosed between a sphere of radius 1 centered on the origin, and a circular cone of half angle 60 degrees with its vertex at the origin as shown in Fig.2. (10 %)

(b). Using Gauss' theorem, find the k value (5 %)

$$\int_V \nabla \cdot \mathbf{r} dV = \int_S \mathbf{r} \cdot d\mathbf{S} = k \int_V dV.$$

(c). Given

$$\mathbf{F} = \frac{\mathbf{r}}{(r^2 + a^2)^{3/2}},$$

find $\nabla \cdot \mathbf{F}$. (5 %) Find the volume integral $\int_V \nabla \cdot \mathbf{F} dV$ (10 %) and surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$, (10 %) where V is the volume of sphere $|\mathbf{r}| = \sqrt{3}a$ and \mathbf{S} is the surface vector on the volume V .

Fig.1

Fig.2

4 Stokes' theorem: $\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \oint \mathbf{a} \cdot d\mathbf{r}$

Given $\mathbf{a} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, $S : x^2 + y^2 + z^2 = a^2, z > 0$ and $C : x^2 + y^2 = a^2, z = 0$, verify Stokes' theorem by determining $\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S}$ and $\oint \mathbf{a} \cdot d\mathbf{r}$. (10 %) Also, calculate $\int_S d\mathbf{S} = \oint_C \mathbf{r} \times d\mathbf{r}$. (10 %)

5 Please explain the relationship among Green's, Stokes' and Gauss' theorems. (10 %)