國立臺灣海洋大學河海工程學系 2000 工程數學 (一) 第三次大考解答

1 The vector field $\mathbf{a} = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2xz)\mathbf{k}$. Show that $\mathbf{a}$ is conservative field and find the line integral of $\int \mathbf{a} \cdot d\mathbf{r}$ along any line joining $(1, 1, 1)$ and $(1, 2, 2)$. (10 %)
Ans: Yes, 11.
2 Green's theorem : $\int\int_A (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \oint_C (Pdx + Qdy)$
Determine the following line integrals and area integrals as shown in Fig.1 (25 %)
(a). $\frac{1}{2}\{\oint_C \{-ydx + xdy\}$, (b). $\frac{1}{2}\oint_C \mid \mathbf{r} \times \hat{\mathbf{t}} \mid ds$, (c). $\frac{1}{2}\oint_C \mathbf{r} \cdot \hat{\mathbf{n}}\, ds$, (d). $\int_A (\nabla \times \mathbf{a}) \cdot \mathbf{k} dA$, and (e). $\int_A dA$, where C is the circular boundary of a unit circle, $\hat{\mathbf{t}}$ is the unit tangent vector, $\hat{\mathbf{n}}$ is the unit normal vector, $(ds)^2 = d\mathbf{r} \cdot d\mathbf{r}$, $\mathbf{a} = (-y, x)$ and A is the area of a unit circle.
Ans: (a). π, (b). π, (c). π, (d). 2π, (e). π.
3 Gauss' theorem: $\int\int\int_V \nabla \cdot \mathbf{a}\, dV = \int\int_S \mathbf{a} \cdot d\mathbf{S}$
(a). Find the volume enclosed between a sphere of radius 1 centered on the origin, and a circular cone of half angle 60 degrees with its vertex at the origin as shown in Fig.2. (10 %)
(b). Using Gauss' theorem, find the k value (5 %)

$$\int_V \nabla \cdot \mathbf{r}\, dV = \int_S \mathbf{r} \cdot d\mathbf{S} = k \int_V dV.$$

(c). Given

$$\mathbf{F} = \frac{\mathbf{r}}{(r^2 + a^2)^{3/2}},$$

find $\nabla \cdot \mathbf{F}$. (5 %) Find the volume integral $\int_V \nabla \cdot \mathbf{F} dV$ (10 %) and surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$, (10 %) where V is the volume of sphere $\mid \mathbf{r} \mid = \sqrt{3}a$ and $\mathbf{S}$ is the surface vector on the volume V.
Ans: (a). $\pi/3$, (b). $k = 3$, (c). $\frac{3a^2}{(r^2+a^2)^{5/2}}$, $3\sqrt{3}\pi/2$.
4 Stokes' theorem: $\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \oint \mathbf{a} \cdot d\mathbf{r}$
Given $\mathbf{a} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, $S : x^2 + y^2 + z^2 = a^2, z > 0$ and $C : x^2 + y^2 = a^2, z = 0$, verify Stokes' theorem by determining $\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S}$ and $\oint \mathbf{a} \cdot d\mathbf{r}$. (10 %) Also, calculate $\int_S d\mathbf{S} = \oint_C \mathbf{r} \times d\mathbf{r}$. (10 %)
Ans: $-2\pi a^2$. $2\pi\mathbf{k}$.
5 Please explain the relationship among Green's, Stokes' and Gauss' theorems. (10 %)
Ans: Green's theorem can derived from Gauss' and Stokes' theorems, respectively.