Then $y = v^{-1/2}$ and

$$y'(x) = -\frac{1}{2}v^{-3/2}v'(x),$$

so the differential equation becomes

$$-\frac{1}{2}v^{-3/2}v'(x) + \frac{1}{x}v^{-1/2} = 3x^2v^{-3/2},$$

or, upon multiplying by $-2v^{3/2}$,

$$v' - \frac{2}{x}v = -6x^2,$$

a linear equation. An integrating factor is $e^{-\int (2/x)\,dx} = x^{-2}$. Multiply the last equation by this factor to get

$$x^{-2}v' - 2x^{-3}v = -6,$$

which is

$$(x^{-2}v)' = -6.$$

Integrate to get

$$x^{-2}v = -6x + C,$$

so

$$v = -6x^3 + Cx^2.$$

The general solution of the Bernoulli equation is

$$y(x) = \frac{1}{\sqrt{v(x)}} = \frac{1}{\sqrt{Cx^2 - 6x^3}}. \blacksquare$$

1.6.3 The Riccati Equation

P. V. ONeil, 2007

DEFINITION 1.8

A differential equation of the form

$$y' = P(x)y^2 + Q(x)y + R(x)$$

is called a *Riccati equation.*

A Riccati equation is linear exactly when $P(x)$ is identically zero. If we can somehow obtain one solution $S(x)$ of a Riccati equation, then the change of variables

$$y = S(x) + \frac{1}{z}$$

transforms the Riccati equation to a linear equation. The strategy is to find the general solution of this linear equation and from it produce the general solution of the original Riccati equation.

EXAMPLE 1.28

Consider the Riccati equation

$$y' = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}.$$

By inspection, $y = S(x) = 1$ is one solution. Define a new variable z by putting

$$y = 1 + \frac{1}{z}.$$

Then

$$y' = -\frac{1}{z^2}z'.$$

Substitute these into the Riccati equation to get

$$-\frac{1}{z^2}z' = \frac{1}{x}\left(1+\frac{1}{z}\right)^2 + \frac{1}{x}\left(1+\frac{1}{z}\right) - \frac{2}{x},$$

or

$$z' + \frac{3}{x}z = -\frac{1}{x}.$$

This is linear. An integrating factor is $e^{\int (3/x)\,dx} = x^3$. Multiply by x^3 to get

$$x^3z' + 3x^2z = (x^3z)' = -x^2.$$

Integrate to get

$$x^3z = -\frac{1}{3}x^3 + C,$$

so

$$z(x) = -\frac{1}{3} + \frac{C}{x^3}.$$

The general solution of the Riccati equation is

$$y(x) = 1 + \frac{1}{z(x)} = 1 + \frac{1}{-1/3 + C/x^3}.$$

This solution can also be written

$$y(x) = \frac{K + 2x^3}{K - x^3},$$

in which $K = 3C$ is an arbitrary constant. ■