

Since  $y_1$  and  $y_2$  are complementary solutions for ODE, we have

$$y_1''(x) + a(x)y_1'(x) + b(x)y_1(x) = 0 \quad (1)$$

$$y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) = 0 \quad (2)$$

Eq.(1)  $\times y_2$  - Eq.(2)  $\times y_1$ , we have

$$\frac{d}{dx} \{y_1 y_2' - y_1' y_2\} + a(x) \{y_1 y_2' - y_1' y_2\} = 0$$

By setting the Wronskian

$$W(x) = W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$W(x)$  satisfies the following first ODE

$$W'(x) + a(x)W(x) = 0$$

The solution is

$$W(x) = k e^{-\int a(x) dx}$$

Without loss of generality, given two degrees of freedoms,  $u_1$  and  $u_2$ , must be determined. By setting the first constraint,

$$u_1' y_1 + u_2' y_2 = c \quad (3)$$

we have

$$u_1' y_1' + u_2' y_2' = f(x) - a(x) c \quad (4)$$

where  $c$  is arbitrary constant. Two equations are summarized

$$y_1 u_1' + y_2 u_2' = c \quad (5)$$

$$y_1' u_1' + y_2' u_2' = f(x) - a(x) c \quad (6)$$

Solve  $u_1'$  and  $u_2'$  first, we have

$$u_1' = \frac{-f(x)y_2}{W(y_1, y_2)} + c \left( \frac{y_2' + ay_2}{W} \right)$$

$$u_2' = \frac{f(x)y_1}{W(y_1, y_2)} - c \left( \frac{y_1' + ay_1}{W} \right)$$

The two additional terms containing  $c$  are present. It is interesting to find that

$$u_1' = c \left( \frac{y_2' + ay_2}{W} \right) \left( \frac{e^{\int a(x) dx}}{e^{\int a(x) dx}} \right) = c (y_2 e^{\int a(x) dx})'$$

$$u_2'; c \left( \frac{y_1' + ay_1}{W} \right) \left( \frac{e^{\int a(x) dx}}{e^{\int a(x) dx}} \right) = c (y_1 e^{\int a(x) dx})'$$

They can be cancelled each other.