THIRD EDITION

MATHEMATICAL METHODS FOR PHYSICS AND ENGINEERING

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frequency corresponds to a solution where the string and rod are moving with opposite phase and $x_1 : x_2 = 9.359 : -16.718$. The two situations are shown in figure 9.1.

In connection with quadratic forms it was shown in section 8.17 how to make a change of coordinates such that the matrix for a particular form becomes diagonal. In exercise 9.6 a method is developed for diagonalising simultaneously two quadratic forms (though the transformation matrix may not be orthogonal). If this process is carried out for A and B in a general system undergoing stable oscillations, the kinetic and potential energies in the new variables η_i take the forms

$$T = \sum_i \mu_i \dot{\eta}_i^2 = \dot{\eta}^{\mathrm{T}} \mathsf{M} \dot{\eta}, \quad \mathsf{M} = \mathrm{diag}\,(\mu_1, \mu_2, \ldots, \mu_N), \tag{9.11}$$

$$V = \sum_i \nu_i \eta_i^2 = \eta^{\mathrm{T}} \mathsf{N} \eta, \quad \mathsf{N} = \mathrm{diag}\,(\nu_1, \nu_2 \ldots, \nu_N), \tag{9.12}$$

and the equations of motion are the *uncoupled* equations

$$\mu_i \ddot{\eta}_i + \nu_i \eta_i = 0, \quad i = 1, 2, \ldots, N. \tag{9.13}$$

Clearly a simple renormalisation of the η_i can be made that reduces all the μ_i in (9.11) to unity. When this is done the variables so formed are called *normal coordinates* and equations (9.13) the *normal equations*.

When a system is executing one of these simple harmonic motions it is said to be in a *normal mode*, and once started in such a mode it will repeat its motion exactly after each interval of $2\pi/\omega_i$. Any arbitrary motion of the system may be written as a superposition of the normal modes, and each component mode will execute harmonic motion with the corresponding eigenfrequency; however, unless by chance the eigenfrequencies are in integer relationship, the system will never return to its initial configuration after any finite time interval.

As a second example we will consider a number of masses coupled together by springs. For this type of situation the potential and kinetic energies are automatically quadratic functions of the coordinates and their derivatives, provided the elastic limits of the springs are not exceeded, and the oscillations do not have to be vanishingly small for the analysis to be valid.

► *Find the normal frequencies and modes of oscillation of three particles of masses m, μm, m connected in that order in a straight line by two equal light springs of force constant k. This arrangement could serve as a model for some linear molecules, e.g.* CO_2.

The situation is shown in figure 9.2; the coordinates of the particles, x_1, x_2, x_3, are measured from their equilibrium positions, at which the springs are neither extended nor compressed.

The kinetic energy of the system is simply

$$T = \tfrac{1}{2} m \left(\dot{x}_1^2 + \mu \dot{x}_2^2 + \dot{x}_3^2 \right),$$

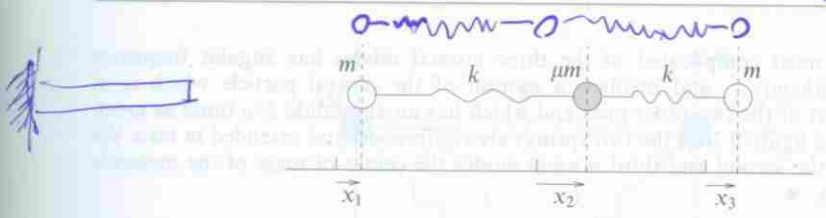

Figure 9.2 Three masses m, μm and m connected by two equal light springs of force constant k.

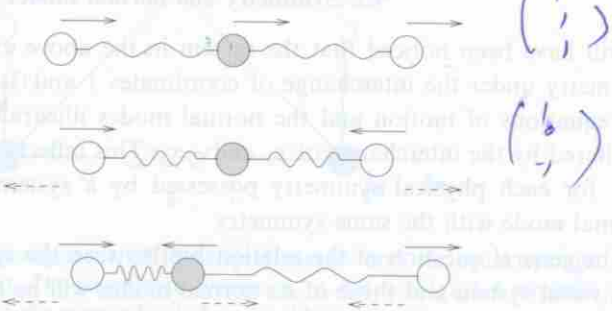

Figure 9.3 The normal modes of the masses and springs of a linear molecule such as CO_2. (a) $\omega^2 = 0$; (b) $\omega^2 = k/m$; (c) $\omega^2 = [(\mu+2)/\mu](k/m)$.

whilst the potential energy stored in the springs is

$$V = \tfrac{1}{2}k\left[(x_2 - x_1)^2 + (x_3 - x_2)^2\right].$$

The kinetic- and potential-energy symmetric matrices are thus

$$\mathsf{A} = \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{B} = \frac{k}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

From (9.10), to find the normal frequencies we have to solve $|\mathsf{B} - \omega^2\mathsf{A}| = 0$. Thus, writing $m\omega^2/k = \lambda$, we have

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\mu\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0,$$

which leads to $\lambda = 0$, 1 or $1 + 2/\mu$. The corresponding eigenvectors are respectively

$$\mathsf{x}^1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathsf{x}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathsf{x}^3 = \frac{1}{\sqrt{2 + (4/\mu^2)}}\begin{pmatrix} 1 \\ -2/\mu \\ 1 \end{pmatrix}.$$

The physical motions associated with these normal modes are illustrated in figure 9.3. The first, with $\lambda = \omega = 0$ and all the x_i equal, merely describes bodily translation of the whole system, with no (i.e. zero-frequency) internal oscillations.

In the second solution the central particle remains stationary, $x_2 = 0$, whilst the other two oscillate with equal amplitudes in antiphase with each other. This motion, which has frequency $\omega = (k/m)^{1/2}$, is illustrated in figure 9.3(b).

The final and most complicated of the three normal modes has angular frequency $\omega = \{[(\mu + 2)/\mu](k/m)\}^{1/2}$, and involves a motion of the central particle which is in antiphase with that of the two outer ones and which has an amplitude $2/\mu$ times as great. In this motion (see figure 9.3(c)) the two springs are compressed and extended in turn. We also note that in the second and third normal modes the centre of mass of the molecule remains stationary. ◄

9.2 Symmetry and normal modes

It will have been noticed that the system in the above example has an obvious symmetry under the interchange of coordinates 1 and 3: the matrices A and B, the equations of motion and the normal modes illustrated in figure 9.3 are all unaltered by the interchange of x_1 and $-x_3$. This reflects the more general result that for each physical symmetry possessed by a system, there is at least one normal mode with the same symmetry.

The general question of the relationship between the symmetries possessed by a physical system and those of its normal modes will be taken up more formally in chapter 29 where the representation theory of groups is considered. However, we can show here how an appreciation of a system's symmetry properties will sometimes allow its normal modes to be guessed (and then verified), something that is particularly helpful if the number of coordinates involved is greater than two and the corresponding eigenvalue equation (9.10) is a cubic or higher-degree polynomial equation.

Consider the problem of determining the normal modes of a system consisting of four equal masses M at the corners of a square of side $2L$, each pair of masses being connected by a light spring of modulus k that is unstretched in the equilibrium situation. As shown in figure 9.4, we introduce Cartesian coordinates x_n, y_n, with $n = 1, 2, 3, 4$, for the positions of the masses and denote their displacements from their equilibrium positions $\mathbf{R}_n$ by $\mathbf{q}_n = x_n\mathbf{i} + y_n\mathbf{j}$. Thus

$$\mathbf{r}_n = \mathbf{R}_n + \mathbf{q}_n \quad \text{with} \quad \mathbf{R}_n = \pm L\mathbf{i} \pm L\mathbf{j}.$$

The coordinates for the system are thus $x_1, y_1, x_2, \ldots, y_4$ and the kinetic energy matrix A is given trivially by $M\mathsf{I}_8$, where I_8 is the 8×8 identity matrix.

The potential energy matrix B is much more difficult to calculate and involves, for each pair of values m, n, evaluating the quadratic approximation to the expression

$$b_{mn} = \tfrac{1}{2}k\left(|\mathbf{r}_m - \mathbf{r}_n| - |\mathbf{R}_m - \mathbf{R}_n|\right)^2.$$

Expressing each $\mathbf{r}_i$ in terms of $\mathbf{q}_i$ and $\mathbf{R}_i$ and making the normal assumption that