(1)G.E

$$x''(t)+\omega^2 x(t)=\sin\varpi t$$

ω：natural frequency of system

ϖ：exciting frequency

I.C: $x(0)=0$ $x'(0)=0$

Taking Laplace transform, we have

$$X(s)=\frac{1}{s^2+\omega^2}\frac{\varpi}{s^2+\varpi^2}$$

$$X(s)=\frac{as+b}{s^2+\omega^2}+\frac{cs+d}{s^2+\varpi^2}$$

決定四個未定係數(a,b,c,d)

$$\begin{cases} a+c=0 \\ b+d=0 \\ a\omega^2+b\varpi^2=0 \\ b\omega^2+d\varpi^2=\varpi \end{cases}$$

$$\Rightarrow a=0\ ,\ c=0\ ,\ b=\frac{-\varpi}{\varpi^2-\omega^2}\ ,\ d=\frac{\varpi}{\varpi^2-\omega^2}$$

$$\therefore X(s)=\frac{\frac{-\varpi}{\varpi^2-\omega^2}}{s^2+\varpi^2}+\frac{\frac{\varpi}{\varpi^2-\omega^2}\frac{1}{\omega}\omega}{s^2+\omega^2}$$

(2)

$$x(t)=\frac{-1}{\varpi^2-\omega^2}\sin\varpi t+\frac{\frac{\varpi}{\omega}}{\varpi^2-\omega^2}\sin\omega t$$

$$=\frac{1}{\omega(\varpi^2-\omega^2)}[\varpi\sin\omega t-\omega\sin\varpi t]$$

(3)

$\omega\to\varpi$ set $\varpi=\omega+\varepsilon$

$$\therefore x(t)=\frac{1}{\omega}\cdot\frac{1}{(\varpi+\omega)(\varpi-\omega)}[(\omega+\varepsilon)\sin\omega t-\omega\sin(\omega t+\varepsilon t)]$$

$$=\frac{1}{\omega}\cdot\frac{1}{\varepsilon\cdot 2\omega}[(\omega+\varepsilon)\sin\omega t-\omega\sin\omega t\cdot\cos\varepsilon t-\omega\cos\omega t\cdot\sin\varepsilon t]$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\omega^2}\cdot\frac{1}{\varepsilon}[\varepsilon\sin\omega t-\omega\sin\omega t\cdot\sin\varepsilon t]$$

$$=\frac{1}{2\omega^2}\cdot[\sin\omega t-\omega t\cos\omega t]$$

(4)check $x(0)=0$ $x'(0)=0$

滿足初始條件，亦滿足非齊次微分方程式，即合外力項之解，故為全解。

(5)

Case1:

$$\varpi \neq \omega \rightarrow x(t)=\frac{1}{\omega}\cdot\frac{1}{(\varpi+\omega)(\varpi-\omega)}[\varpi\sin\omega t-\omega\sin\varpi t]$$

Case2:Beating

$$\varpi \approx \omega \rightarrow x(t)=\lim_{\substack{\varepsilon\to 0\\ \varepsilon\neq 0}}\frac{1}{2\omega^2}\cdot\frac{1}{\varepsilon}[\varepsilon\sin\omega t-\omega\sin\omega t\cdot\sin\varepsilon t]$$

$$=\frac{1}{2\omega^2}\cdot[\sin\omega t-\omega t\cos\omega t]$$

Case3:Resonance

$$\varpi = \omega \rightarrow x(t)=\frac{1}{2\omega^2}\cdot[\sin\omega t-\omega t\cos\omega t]$$

Case1: $\omega = 2\pi$, $\varpi = 4\pi$

Case2: $\omega = 2\pi$, $\varpi = 1.99\pi$

Case3: $\omega = 2\pi$, $\varpi = 2\pi$

(6)Set $\varpi = \omega$, Solve by Laplace transform

$$x(t)=\frac{\omega}{(s^2+\omega^2)^2}=\frac{s^2+\omega^2-s^2}{(s^2+\omega^2)^2}\cdot\frac{1}{\omega}$$

$$=\frac{1}{\omega}[\frac{1}{s^2+\omega^2}-\frac{s^2}{(s^2+\omega^2)^2}]$$

Using

$$L[\sin\omega t]=\frac{\omega}{s^2+\omega^2} \qquad L[\cos\omega t]=\frac{s}{s^2+\omega^2}$$

$$L[t\sin\omega t]=\frac{-\omega(2s)}{(s^2+\omega^2)^2}\cdot(-1) \qquad L[t\cos\omega t]=-[\frac{1}{(s^2+\omega^2)}-\frac{2s^2}{(s^2+\omega^2)^2}]$$

⇨ $L^{-1}[\frac{s^2}{(s^2+\omega^2)^2}]=\frac{1}{2}t\cos\omega t+\frac{1}{\omega}\sin\omega t$

⇨

$$x(t)=\{-\frac{1}{2}[t\cos\omega t+\frac{1}{\omega}\sin\omega t]+\frac{1}{\omega}\sin\omega t\}\cdot\frac{1}{\omega}$$
$$=\frac{1}{2\omega^2}[\sin\omega t-\omega t\cos\omega t]$$

和前面假設不同，做出後 *x(t)*，再以ϖ→ω逼近，所得結果相同。

另以迴旋積分(Convolution)求解

$$X(s)=\omega\cdot(\underbrace{\frac{1}{s^2+\omega^2}}_{L[\sin\omega t]})\cdot(\underbrace{\frac{1}{s^2+\omega^2}}_{L[\sin\omega t]})$$

$$x(t)=\frac{1}{\omega}\int_0^t\sin\omega u\cdot\sin\omega(t-u)\,du=\frac{-1}{2\omega}\int_0^t-2\sin\omega u\cdot\sin\omega(t-u)\,du$$
$$-2\sin\alpha\cdot\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$$
$$\therefore\ -2\sin\omega u\cdot\sin\omega(t-u)=\cos\omega t-\cos(2\omega u-\omega t)$$

⇨

$$x(t)=\frac{-1}{2\omega}\int_0^t[\cos\omega t-\cos(2\omega u-\omega t)]\,du$$
$$=\frac{-1}{2\omega}[t\cos\omega t-\underbrace{\int_0^t\cos(2\omega u-\omega t)\,du}_{set\,u=\bar{u}+\frac{t}{2}}]$$

in which

$$-\int_0^t\cos(2\omega u-\omega t)\,du=-\int_{-\frac{t}{2}}^{\frac{t}{2}}\cos2\omega\bar{u}\,d\bar{u}$$
$$=-\frac{1}{2\omega}\sin2\omega\bar{u}\Big|_{-\frac{t}{2}}^{\frac{t}{2}}=-\frac{1}{\omega}\sin\omega t$$

⇨

$$x(t)=\frac{-1}{2\omega}[t\cos\omega t-\frac{1}{\omega}\sin\omega t]$$
$$=\frac{1}{2\omega^2}[\sin\omega t-\omega t\cos\omega t]$$