工數二(B)第一次大考解答(一階 ODE)

1. setting c=0, $y_3 = y_2 + \frac{1}{z} = -2 + \frac{1}{z}$ 代入 Ricatti equation

$$y' = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}$$

$$\left(\frac{1}{z}\right)' = \frac{1}{x}\left(-2+\frac{1}{z}\right)^2 + \frac{1}{x}\left(-2+\frac{1}{z}\right) - \frac{2}{x} = \frac{-3}{xz} + \frac{1}{xz^2}$$

$$z^2\left(\frac{1}{z}\right)' = \frac{-3}{x}z + \frac{1}{x}$$

$$z^2\left(-\frac{1}{z^2}\right)z' = \frac{-3}{x}z + \frac{1}{x} \Longrightarrow z' - \frac{3}{x}z = -\frac{1}{x} \quad \text{(Linear equation)}$$

Integrating factor, I(x) $= e^{\int \frac{-3}{x}dx} = x^{-3}$

I(x) z $= \int x^{-3}\left(-x^{-1}\right)dx = \frac{x^{-3}}{3} + c_1$

z $= \frac{x^{-3}+c}{3x^{-3}}$

$$y_3 = y_2 + \frac{1}{z} = -2 + \frac{3x^{-3}}{x^{-3}+c} = -2 + \frac{3}{1+cx^3}$$

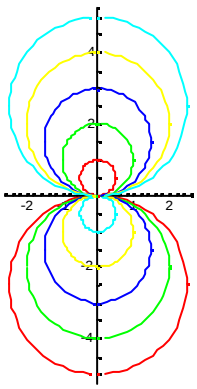
$let\ c = 0, y_3 = 1$

2. $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

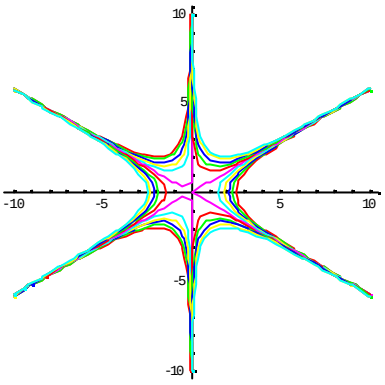
(1) <u>Not Separable</u>. <u>Not Exact</u>. <u>Homogeneous</u>.

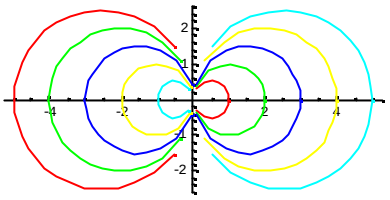
(2) $I = y^{-2}$.

(3) $z_1(x,\ y) = \frac{x^2}{y} + y = c$

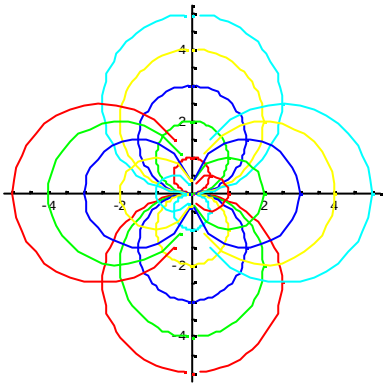


(4) $z_2(x,\ y) = xy^2 - \frac{1}{3}x^3 = c$ $\qquad$ $z_2(x,\ y) = \frac{y^2}{x} + x = c$ $\quad \frac{dy}{dx} = \frac{x^2 - y^2}{-2xy}$





(5) Please compare figures in (4) and (5), and explain.



Orthogonal

3. $\frac{dy}{dx} + \frac{2}{x}y = x^2$

$$I = e^{\int \frac{2}{x}dx} = x^2$$

$$x^2\frac{dy}{dx} + 2xy = x^4$$

$$(x^2 y)' = x^4$$

$$x^2 y = \frac{1}{5}x^5 + c$$

$$y = \frac{1}{5}x^3 + cx^{-2}$$

4.

(法一)

$16y_1'' = 2y_1 \cdot 9.81$

$16\dfrac{dy_1'}{dt} = 2y_1 \cdot 9.81$

$\dfrac{dy_1'}{dt} = 1.23y_1 \quad \Rightarrow \quad \dfrac{dy_1'}{dy_1}\dfrac{dy_1}{dt} = 1.23y_1$

$(y_1')^2 = 1.23y_1^2 + c$

when $y_1(0) = 1$ and $y_1'(0) = 0$

$c = -1.23$

$(y_1')^2 = 1.23(y_1^2 - 1)$

$y_1' = 1.11\sqrt{y_1^2 - 1}$

(1) $t = \int dt = \int_1^8 \dfrac{dt}{dy}dy = \int_1^8 \dfrac{1}{1.11\sqrt{y_1^2 - 1}}dy$

$t = \dfrac{1}{1.11}(\ln|x + \sqrt{x^2 - 1}|)\,|_1^8 = 2.50$ (sec)

Yes.

(2) when cable escapes the roller ($y_1 = 8$), $y_1' = 8.80$ (m/s)

No.

(3) $16y_2'' = 2y_2 \cdot 9.81 + 19.62$

(4) $y_2(0) = 0$

(法二)

Let $y = e^{st}$

$16y_1'' = 2y_1 \cdot 9.81$

$(8s^2 - 9.81)e^{st} = 0$

$s = \pm 1.11$

$y_1 = c_1 e^{1.11t} + c_2 e^{-1.11t}$

when $y_1(0) = 1$ and $y_1'(0) = 0$

$c_1 = c_2 = 0.5$

$y_1 = \dfrac{1}{2}e^{1.11t} + \dfrac{1}{2}e^{-1.11t} = \cosh(1.11t)$

when cable escapes the roller ($y_1 = 8$)

$t = \dfrac{\cosh^{-1}(8)}{1.11} = 2.50$ (sec)

$y_1' = \sinh(1.11t) = \sinh(1.11 \cdot 2.50) = 8.80$ (m/sec)

5. $let\ y' = p$ 代入 $y = xy' + y'^2$ 整理得 $p'(2p + x) = 0$

case1 $p' = 0$

$y'' = 0 \Longrightarrow y = ax + b$ 代入原式 $\Longrightarrow y = ax + a^2$

case2 $2p + x = 0$

$2y' + x = 0 \Longrightarrow y = -\frac{1}{4}x^2 + c$ 代入原式 $\Longrightarrow y = -\frac{1}{4}x^2$

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