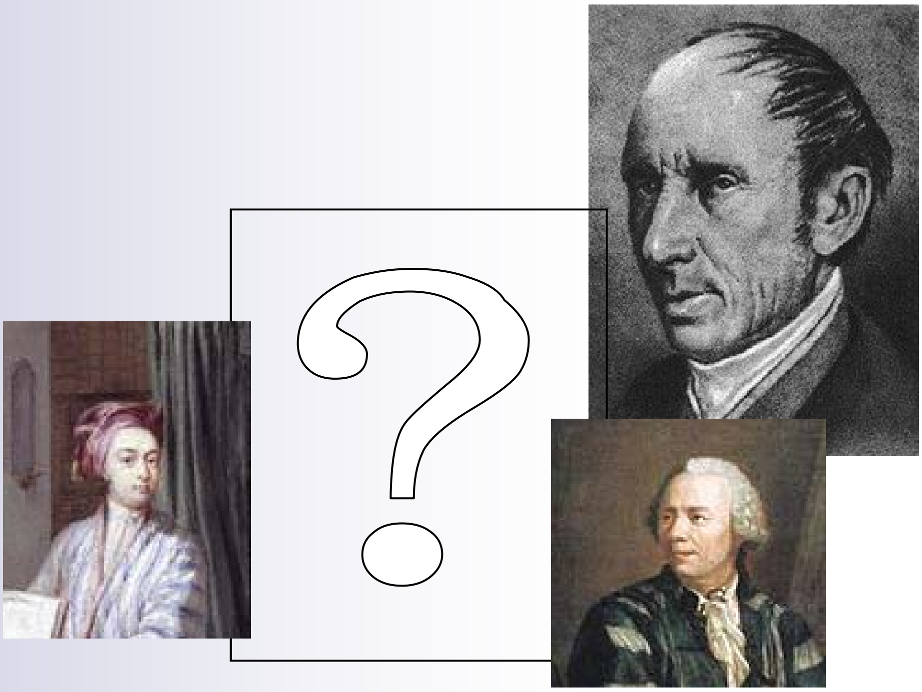
工程數學(一) (2011)



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力學聲響振動實驗室　工程數學(一)2011,09~2012,01　陳正宗終身特聘教授

第零章 工程數學修課相關資料

授 課 課 程 大 綱

課程名稱：	工程數學(一) Engineering Mathematics (I)
教 師：	陳正宗
課程目標：	本課程宗旨為訓練學生熟悉數學思考的方法，以便在往後的學習、工作、研究或實際工程問題多提供想像與思考的空間，並和物理觀念一氣呵成，即能掌握理論之精義，並實際感受數學模式對物理現象的詮釋，強調工程數學是一門活的學問，是思考問題、感覺体會問題的利器，相信對學生在往後的學術研究、實務工作甚至考試上均有助益。
學 分 數：	3學分(大二必修)
課程內容：	• Vector calculus • Fourier 級數 • Fourier 積分轉換 • Laplace transform
參考書籍：	• P. V. O'Neil, Advanced Engineering Mathematics, Fourth Edition, PWS Publ., Boston, 1995. • C. R. Wylie and L. C. Barrett, Advanced Engineering Mathematics, Sixth Edition, McGRaw-Hill, New York, 1995. • E. Kreyszig, Advanced Engineering Mathematics, Seventh Edition, John Wiley \& Sons, New York, 1993. • S. I. Grossman and W. R. Derrick, Advanced Engineering Mathematics, Seventh Edition, Harper & Row, Cambridge, 1988. • W. Kaplan, Advanced Mathematics for Engineers, Fourth Edition, Addison-Wesley, 1981. • 陳正宗，工程數學講義，海洋大學河工系，基隆，1995.
上課時間：	星期四9:00¯12:00　河二館504
評分標準：	作業10%、四次小考40%、期中考20%、期末考20%、平時表現10%。
先修課程：	大一微積分 工數(一)

國立臺灣海洋大學100學年度第1學期行事曆

（99學年度第2學期第2次行政會議通過；教育部100年4月28日臺高（一）字第1000068100號函同意備查）

年	月	星期／週次	星期日	星期一	星期二	星期三	星期四	星期五	星期六	舉　辦　事　項
中華民國一○○年	八月			1	2	3	4	5	6	（1）第一學期開始，99暑修第二期開始上課，就學貸款申辦開始日（1-4）研究所新生住宿電腦抽籤申請
			7	8	9	10	11	12	13	（12）復學生註冊（13）新生入學說明會暨新生家長日
			14	15	16	17	18	19	20	（15-22）大學部新生住宿申請（15-26）申請抵免學分
			21	22	23	24	25	26	27	
			28	29	30	31				（28）祖父母節
	九月						1	2	3	（3）暑假結束（3-4）學生宿舍開放
			4	5	6	7	8	9	10	（5）日間學制研究生新生及進修學士班、碩專班新生體檢；進修學士班、碩專班新生註冊及就學貸款申辦截止日；進修學士班、碩專班新生入學教育（6）日間學制研究生新生註冊暨選課說明會及就學貸款申辦截止日；日間學制學士班新生體檢（7）日間學制學士班新生註冊暨選課說明會及就學貸款申辦截止日（8）新生安全日（8-9）日間學制新生入學教育（8-16）1001 舊生第三階段電腦選課作業，新生電腦選課作業（9/5-10/23）弱勢助學計畫申請
		一	11	12	13	14	15	16	17	（11）各學制學生開始上課（12）中秋節（放假）（13-14）各學制舊生註冊（14）舊生就學貸款申辦截止日（15-16）准假學生補註冊（17-23）人工特殊加選及超修學分申請（9/13-10/14）校內獎學金申請
		二	18	19	20	21	22	23	24	
		三	25	26	27	28	29	30		（30）選課上網確認截止日
	十月								1	10月上旬預官考選報名作業
		四	2	3	4	5	6	7	8	
		五	9	10	11	12	13	14	15	（10）國慶日(放假)（15）本校五十八週年校慶（全校正常上班上課），陸上運動會
		六	16	17	18	19	20	21	22	（20-31）碩士生學位考試申請
		七	23	24	25	26	27	28	29	
		八	30	31						
	十一月				1	2	3	4	5	
		九	6	7	8	9	10	11	12	（6-12）期中考試（11）學分費繳交截止日
		十	13	14	15	16	17	18	19	（14）期中預警輸入開始日（14-25）1001期中退選作業
		十一	20	21	22	23	24	25	26	
		十二	27	28	29	30				（28）期中預警輸入截止日（30）博士生學位考試申請截止日（11/28-12/9）下學期學雜費減免申請
	十二月						1	2	3	
		十三	4	5	6	7	8	9	10	（5-9）輔系、雙主修申請（9-15）1002第一階段電腦選課作業
		十四	11	12	13	14	15	16	17	
		十五	18	19	20	21	22	23	24	（21-23）1002電腦選課抽籤作業
		十六	25	26	27	28	29	30	31	
一○一年	一月	十七	1	2	3	4	5	6	7	（1）開國紀念日（適逢例假日，不補假）（2）補假一日(校慶)（3）就學貸款申辦開始日（3-9）1002第二階段電腦選課作業（6）1001學期申請休學截止日
		十八	8	9	10	11	12	13	14	（8-14）學期考試
			15	16	17	18	19	20	21	（15）寒假開始，1月下旬預官考選考試（20）教職員彈性放假（20-29）寒假期間部分宿舍關閉
			22	23	24	25	26	27	28	（22-26）除夕暨春節假期（放假，26為補假）（27）教職員彈性放假
			29	30	31					（31）第一學期結束，研究生論文繳交截止日

國立臺灣海洋大學100學年度第2學期行事曆

（99學年度第2學期第2次行政會議通過；教育部100年4月28日臺高（一）字第1000068100號函同意備查）

年	月	週次 \ 星期	星期日	星期一	星期二	星期三	星期四	星期五	星期六	舉辦事項
中華民國一〇一年	二月					1	2	3	4	（1）第二學期開始
			5	6	7	8	9	10	11	2月中旬預官考選成績發放（9）復學生註冊
			12	13	14	15	16	17	18	（16-24）1002第三階段電腦選課（18）寒假結束
		一	19	20	21	22	23	24	25	（19）各學制學生開始上課（20-21）註冊（21）就學貸款申辦截止日（22-23）准假學生補註冊（2/25-3/2）1002 人工特殊加選及超修學分申請
		二	26	27	28	29				（28）和平紀念日（放假）
	三月						1	2	3	（1-30）校內獎學金申請
		三	4	5	6	7	8	9	10	3月中旬預官選填志願申請，研發替代役申請（9）選課上網確認截止日
		四	11	12	13	14	15	16	17	（12-19）舊生床位保留申請（17）中、英文會考
		五	18	19	20	21	22	23	24	
		六	25	26	27	28	29	30	31	
	四月	七	1	2	3	4	5	6	7	（4）兒童節（放假）（5）民族掃墓節（放假）（6）敦親活動（教職員彈性放假、該日課程由教師自行擇期補課）
		八	8	9	10	11	12	13	14	
		九	15	16	17	18	19	20	21	（15-21）期中考試（20）學分費繳交截止日（20-30）碩士生學位考試申請
		十	22	23	24	25	26	27	28	（23）期中預警輸入開始日（23-27）轉系申請（23-30）舊生住宿電腦抽籤申請（4/23-5/4）期中退選作業，教育學程申請
		十一	29	30						（4/30-5/11）下學期學雜費減免申請
	五月				1	2	3	4	5	（3-4）變更轉系申請
		十二	6	7	8	9	10	11	12	5月中旬預官錄取公告（7）期中預警輸入截止日
		十三	13	14	15	16	17	18	19	（14-18）100暑修第一期登記（18）轉系考試（18-24）1011第一階段電腦選課
		十四	20	21	22	23	24	25	26	（20-26）畢業生學期考試（21-28）學生宿舍暑期住宿申請
		十五	27	28	29	30	31			（5/30-6/1）電腦選課抽籤作業（31）博士生學位考試申請截止日，水上運動會
	六月							1	2	
		十六	3	4	5	6	7	8	9	（9）畢業典禮（全校正常上班上課）
		十七	10	11	12	13	14	15	16	（11）補假一日（畢業典禮）（12-18）1011第二階段電腦選課（15）1002學期申請休學截止日（15-22）研究所新生暑期住宿申請
		十八	17	18	19	20	21	22	23	（17-23）學期考試（23）端午節（適逢例假日，不補假）
			24	25	26	27	28	29	30	（24）暑假開始（29-30）學生宿舍關閉（30）暑期住宿宿舍開放
	七月		1	2	3	4	5	6	7	
			8	9	10	11	12	13	14	（9）100暑修第一期開始上課（9-13）100暑修第二期登記（13-20)研究所新生床位保留申請
			15	16	17	18	19	20	21	
			22	23	24	25	26	27	28	
			29	30	31					（31）第二學期結束，研究生畢業論文繳交截止日

備註：

1、具原住民身分者，其各該原住民族歲時祭儀放假日期，依行政院原住民族委員會公告日期放假。（紀念日及節日實施辦法第4條）

2、國定假日、春節如有調整，請人事室另行公告。

3、選課日程如有調整，由教務處註冊課務組/進修推廣組另行公告。

001	河海工程學系 1B	09952101	黃子翔	
002	河海工程學系 1B	09952102	劉家如	
003	河海工程學系 1B	09952103	余柏宏	
004	河海工程學系 1B	09952104	張永霖	
005	河海工程學系 1B	09952105	張力	
006	河海工程學系 1B	09952106	藍舒平	
007	河海工程學系 1B	09952107	吳旭強	
008	河海工程學系 1B	09952108	陳弘彝	
009	河海工程學系 1B	09952109	邱駿豪	
010	河海工程學系 1B	09952110	陳峙霖	
011	河海工程學系 1B	09952111	邱揚凱	
012	河海工程學系 1B	09952112	黃仲安	
013	河海工程學系 1B	09952113	吳振豪	
014	河海工程學系 1B	09952114	高怡絹	
015	河海工程學系 1B	09952115	王藝潔	
016	河海工程學系 1B	09952116	王昱璁	
017	河海工程學系 1B	09952117	張恩慈	
018	河海工程學系 1B	09952118	郭哲瑋	
019	河海工程學系 1B	09952119	李柏霖	
020	河海工程學系 1B	09952120	謝晴杰	
021	河海工程學系 1B	09952121	李佩蓉	
022	河海工程學系 1B	09952122	顏立翔	
023	河海工程學系 1B	09952123	許少陵	
024	河海工程學系 1B	09952124	賴奕源	
025	河海工程學系 1B	09952125	李佩諭	
026	河海工程學系 1B	09952126	戴　薇	
027	河海工程學系 1B	09952127	林竹恩	
028	河海工程學系 1B	09952128	林翊翔	
029	河海工程學系 1B	09952129	楊晉	
030	河海工程學系 1B	09952130	許哲崙	
031	河海工程學系 1B	09952131	黃昱豪	
032	河海工程學系 1B	09952132	陳振鈞	
033	河海工程學系 1B	09952133	胡韻芳	

034	河海工程學系 1B	09952134	林剛弘	
035	河海工程學系 1B	09952135	張育仁	
036	河海工程學系 1B	09952136	劉宗瑋	
037	河海工程學系 1B	09952137	徐兆緯	
038	河海工程學系 1B	09952138	彭櫳漢	
039	河海工程學系 1B	09952139	文軍強	
040	河海工程學系 1B	09952140	陳堯夫	
041	河海工程學系 1B	09952141	林昱汶	
042	河海工程學系 1B	09952142	詹承諭	
043	河海工程學系 1B	09952143	張哲嘉	
044	河海工程學系 1B	09952144	王思晴	
045	河海工程學系 1B	09952145	劉　慬	
046	河海工程學系 1B	09952146	陳韋翰	
047	河海工程學系 1B	09952147	鄭宇宏	
048	河海工程學系 1B	09952148	楊瑾	
049	河海工程學系 1B	09952149	黃舜揚	
050	河海工程學系 1B	09952150	羅文韋	
051	河海工程學系 1B	09952151	朱佩欣	
052	河海工程學系 2A	09952204	連家琪	
053	河海工程學系 3A	B97520019	陳世航	
054	河海工程學系 3A	B97520032	廖育德	
055	河海工程學系 3A	B97520041	盧關安	
056	河海工程學系 3A	B97520055	張志霖	
057	河海工程學系 3B	B97520121	洪鼎爲	
058	河海工程學系 2A	B98520009	王振信	
059	河海工程學系 2A	B98520018	陳楓霖	
060	河海工程學系 2A	B98520026	莊祐銓	
061	河海工程學系 2A	B98520033	簡高平	
062	河海工程學系 2A	B98520048	黃粲元	
063	河海工程學系 2B	B98520131	王冠智	

寫在前面

本講義係根據本人在海大河工系教授工程數學多年 累積的手稿, 除了包括工數內容的介紹外, 亦含蓋了歷年的作業、小考、大考及一些參考解答。希望對工數有興趣的同學、老師、學者專家或工程師們能有參考的價值。作者將多年來教學的心得與實務工作經驗融入, 並加入習題, 儘量朝向適合當教科書的方向來努力。同時, 也將最近研究過程所用到的工數併入, 希望初學者在學習的過 程中, 除了能入門外, 也能 體會到學工數眞正的目的與樂趣。當然, 我們也希望經由興趣的培養, 能對工數有更深一層的体認, 而 不僅流於應付考試的庸俗想法。本講義的資料, 多數取材於現今幾本知名的工數教科書, 當然也含蓋我們在教學研究多年所 獲得的心得。除了感謝作者的老師、同事、同好、研究生與修 課同學們的貢獻外, 過往學者專家對本講義的 諸多建議, 也一併予以考慮。然而再有的暇疵, 當然是由於作者的疏忽所致。若有任何批評與指教, 煩請告知。

陳正宗 Jeng-Tzong Chen, Ph.D., Prof.
海洋大學河工系教授
E mail:B0209@ntou66.ntou.edu.tw
E mail:jtchen@ind.ntou.edu.tw

西元一九九九年九月二十八日
台北郵政 23 之 36 號信箱
基隆郵政 7 之 59 號信箱

工程數學 (一)-二B 班(Prof. J. T. Chen) 修課與旁聽資料

姓名 學號 系級 選修或旁聽 系所組別 微積分成績 講義購買

海大河工系陳正宗 修課名單參考資料

工程數學(2011)相關課程相關資訊

課名	教師	小老師/助教
工程數學(二)2A	張景鐘 教授	
工程數學(二)2B	陳正宗 終身特聘教授	李應德 李家瑋 范羽 郭柏伸 陳逸維 將立傑 陳衍穎 蔡育仁

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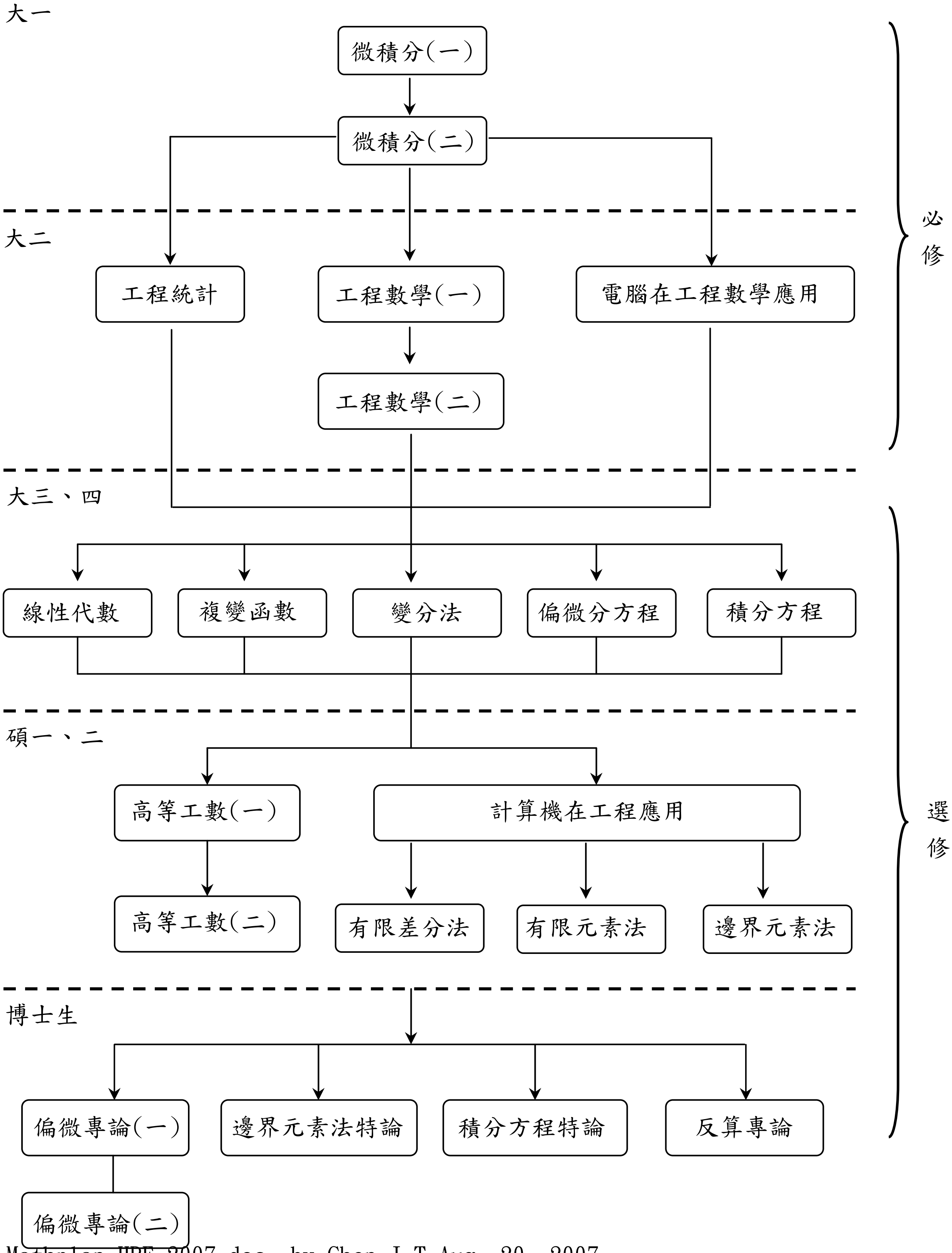
(革命尚未成功，同志仍需努力)->(國父)　　(學業尚未成功，博士仍需努力)->(鮑院士)

(學生尚未聽懂，老師與助教仍需努力)->(陳正宗終身特聘教授與助教)　　(知識尚未充足，學生仍需努力)->(李家瑋)

年　代	教　師	教 課 書/助教	學　生
1994	陳正宗(工數三 2,8、四 21)	Churchill, Farlow(游大偉與楊森翔)	陳桂鴻
1995	梁明德、陳正宗(B)	Grossman(陳桂鴻與翁煥昌)	林建華
1996	梁明德、陳正宗(B)	Whylie (陳鈺文與鄭超明)	李慶鋒
1997	梁明德、葉為忠	Kreyszig	劉立偉
1997	陳正宗(工數三 58、四 19)	Churchill, Farlow(丘宜平與黃川夏)	李慶鋒
1998	梁明德、陳正宗(A)	Greenberg(林建華,程永正,鍾渝隆)	林盛益
1999	林三賢、陳正宗(B)	Kreyszig(李慶鋒,陳韋誌,張銘翰)	葉雅婷
2000	梁明德、陳正宗(B)	Riley(劉立偉與林書睿)	朱雅雯
2001	梁明德、林三賢	Lopez	蔡孟蓉
2001	陳正宗(工數三 42、四 15)	Riley(林盛益與林宗衛)	李應德
2002	陳正宗(B)、張景鐘 、曹登皓	Oneil(吳清森與李應德)	林智凱
2003	陳正宗(B)、張景鐘、曹登皓	Oneil(蕭嘉俊,沈文成,陳佳聰)	林坤生
2004	陳正宗(工數三 68)	Oneil(李應德與高政宏)	吳建和
2004	陳桂鴻(工數一、二) 2A	Oneil(沈文成與吳安傑)	吳國綸
2004	呂學育(工數一、二) 2B	Oneil(蕭嘉俊與陳柏源)	謝紹恆
2005	陳正宗(工數四 23)	Oneil(李應德與陳佳聰)	高聖凱
2005	陳桂鴻(工數一、二) 2A	Zill(高政宏)	林裕桀/余尚儒
2005	呂學育(工數一、二) 2B	Zill(柯佳男與廖奐禎)	曹大發
2006	陳正宗(複變 19)	Brown and Churchil(李應德)	吳國綸
2007	陳正宗(偏微分方程 11)	Riley(廖奐禎與高聖凱)動畫	周克勳
2007	葉為忠(2A) 陳正宗(2B/65)	Kreyszig(周克勳與李家瑋)	陳聖詒/賴芝亭
2007	徐文信(複變)	Churchill	林羿州/謝祥志
2011	陳正宗(B)、張景鐘、曹登皓	Kreyszig(范 羽、陳逸維與郭柏伸)	李佩諭

(File: book-2011-math1.doc, date: Jun.28/2011)　2005 08-2006 07 休假

海大河工系數學相關課程規劃　　陳正宗終身特聘教授



Mathplan-HRE-2007.doc by Chen J T Aug. 20, 2007.

數理程度 每下愈況 大家加油

海洋大學工學院基礎數學課程教學內容

微積分(上): 1. 函數、極限與連續函數 2. 微分的運算與基本函數的微分 3. 微分的性質與應用 4. 定積分、不定積分與微積分基本定理 5. 積分應用 6. 超越函數。

微積分(下): 1. 超越函數 2. 積分技巧 3. 瑕積分 4. 無窮級數 5. 多變數函數微分 6. 極坐標 7. 重積分。

工程數學(一): 1. 一階常微分方程 2. 高階常微分方程 3. 矩陣。

工程數學(二): 1. 向量分析 2. Fourier級數及轉換 3. Laplace轉換。

附件五之一

應用力學科目分類

Ⅰ. FOUNDATIONS AND BASIC METHODS 力學基礎與基本方法

100 Continuum mechanics 連體力學
102 Finite element methods 有限單元法
104 Finite difference methods 有限差分法
106 Other computational methods 其他計算方法
108 Modeling 理論模式
110 Experimental system analysis 實驗系統分析

Ⅱ. DYNAMICS AND VIBRATION 動力學與振動學

150 Kinematics and dynamics 運動學及動力學
152 Vibrations of solids (basic) 固體振動學(基本)
154 Vibrations (structural elements) 振動學(結構單元)
156 Vibrations (structures) 振動學(結構)
158 Wave motions in solids 固體中之波動學
160 Impact on solids 撞擊與衝擊
162 Waves in incompressible fluids 不可壓縮流中之波動學
164 Waves in compressible fluids 可壓縮流中之波動學
166 Solid-fluid interactions 固體與流體之互制作用
168 Astronautics (celestial, orbital mechanics)
太空學(天體軌道力學)
170 Explosions and ballistics 爆炸與彈道學
172 Acoustics 聲學

Ⅲ. AUTOMATIC CONTROL 自動控制

200 Syetems theory and design 系統理論與設計
202 Optimal control systems 優選控制系統
204 Systems and control applications 系統與控制應用
206 Robotics 機器人學
208 Manufacturing 製造學

Ⅳ. MECHANICS OF SOLIDS 固體力學

250 Elasticity 彈性力學
252 Viscoelasticity 黏彈性力學
254 Plasticity and viscoplasticity 塑性力學與黏塑性力學
256 Composite material mechanics 複合材料力學
258 Cables, ropes, beams, etc. 纜、索、樑
260 Plates, shells, membranes, etc. 板、殼、膜
262 Structural stability (buckling, postbuckling)
結構穩定學(挫屈、後挫屈)
264 Electromagneto-solid mechanics 電磁固體力學
266 Soil mechanics (basic) 土壤力學(基本)
268 Soil mechanics (applied) 土壤力學(應用)
270 Rock mechanics 岩石力學
272 Materials processing 材料製程
274 Fracture and damage processes 破裂與損傷過程
276 Fracture and damage mechanics 破裂與損傷力學
278 Experimental stress analysis 實驗應力分析
280 Material testing 材料測試
282 Structures (basic) 結構學(基本)
284 Structures (ground) 結構學(地面結構)
286 Structures (ocean and coastal) 結構學(海洋與海岸)
288 Structures (mobile) 結構學(移動)
290 Structures (containment) 結構學(覆蓋保護)
292 Friction and wear 摩擦學
294 Machine elements 機械單元
296 Machine design 機械設計
298 Fastening and joining 連接與固定

Ⅴ. MECHANICS OF FLUIDS 流體力學

350 Rheology 流變學
352 Hydraulics 水力學
354 Incompressible flow 不可壓縮流
356 Compressible flow 可壓縮流
358 Rarefied flow 稀薄流
360 Multiphase flows 多相流
362 Wall layers (including boundary layers) 壁層(包括邊界層)
364 Internal flow (pipe, channel, and Couette) 內流(管、槽等)
366 Internal flow (inlets, nozzles, diffusers, and cascades)
內流(導管、噴嘴、擴散翼列)
368 Free shear layers (mixing layers, jets, wakes, cavities, and plumes) 自由剪變流(混合層、射注、尾流、空泡和煙柱)
370 Flow stability 流體穩定學
372 Turbulences 紊流
374 Electromagneto-fluid and plasma dynamics 電磁流體與電漿動力學
376 Naval hydromechanics 船艦流體動力學
378 Aerodynamics 空氣動力學
380 Machinery fluid dynamics 機械流體動力學
382 Lubrication 潤滑學
384 Flow mesasurements and visualization 流體量測與觀測

Ⅵ. THERMAL SCIENCES 熱學

400 Thermodynamics 熱力學
402 Heat transfer (one phase convection) 熱傳遞(單相對流)
404 Heat transfer (two phase convection) 熱傳遞(雙相對流)
406 Heat transfer (conduction) 熱傳遞(傳導)
408 Heat transfer (radiation, combined modes)
熱傳遞(輻射、混合模態)
410 Heat transfer (devices and systems) 熱傳遞(儀器與系統)
412 Thermomechanics of solids 固體熱力學
414 Mass transfer (with or without heat transfer)
質量傳遞(含及不含熱傳遞)
416 Combustion 燃燒學
418 Prime movers and propulsion systems 原動力機與推進系統

Ⅶ. EARTH SCIENCES 地球科學

450 Micromeritics 微粒學
452 Porous media 孔隙介質
454 Geomechanics 大地力學
456 Earthquake mechanics 地震力學
458 Hydrology, oceanology, and meteorology 水文學、海洋學、氣象學

Ⅷ. ENERGY AND ENVIRONMENT 能源與環境

500 Fossil fuel systems 礦業燃料系統
502 Nuclear systems 核能系統
504 Geothermal systems 地熱系統
506 Solar systems 太陽能系統
508 Wind energy systems 風能系統
510 Ocean energy systems 海洋能系統
512 Energy distribution and storage 能源分送與貯藏
514 Environmental fluid mechanics 環境流體力學
516 Hazardous waste containment and disposal 公害廢物處理

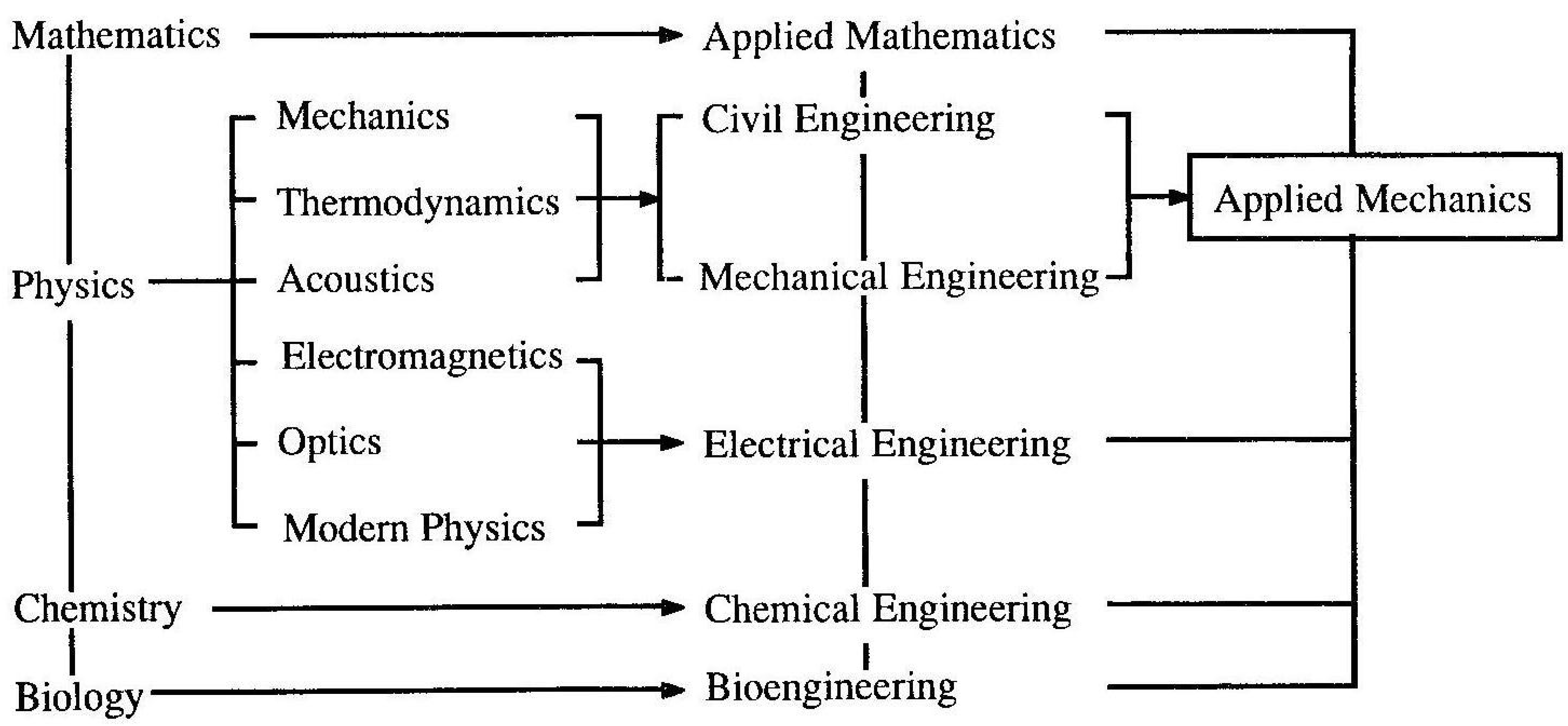
Ⅸ. BIOSCIENCES 生物科學

550 Biomechanics 生物力學
552 Human an factors engineering 人性因素與工程
554 Rehabilitation engineering 復建工程
556 Sports mechanics 運動力學

Ⅹ. GENERAL AND MISCELLANEOUS 其他

(錄自美國機械工程學會「Applied Mechanics Reviews-1989」ASME)

RELATIONSHIPS AMONG APPLIED MECHANICS, BASIC SCIENCE, AND ENGINEERING



工程數學 2B 班(2011,9-2012,1)上課日程及內容

陳正宗終身特聘教授(海大河工系)E-mail:jtchen@mail.ntou.edu.tw

週數	日期	預定進度		備註
1	Sep.14	Introduction		檢定考試
2	Sep.21	ODE(order 1).	Physics	小考(1)
3	Sep.28	ODE(order 1)	Model	GE, IC and BC
4	Oct.05	ODE(order 1)	Separable form	小考(2)
5	Oct.12	ODE(order 1)	Exact form	小考(3)
6	Oct.19	ODE(order 1)	Homogeneous form	作業 Linear independence
7	Oct.26	ODE(order 1)	Special form	期中考
8	Nov.02	ODE(order 2)	Comp. & particular	
9	Nov.09	ODE(order 2)	Beating resonance	小考(4)
10	Nov.16	ODE(higher order)	Frame indifference	
11	Nov.23	ODE(higher order)	Euler and Cauchy	小考(5)
12	Nov.30	ODE(higher order)		
13	Dec.07	Vector and tensor	.moment of inertia	
14	Dec.14	Vector and tensor	Mohr circle	
15	Dec.21	Vector and tensor		
16	Dec.28	Matrix		
17	Jan.04	Matrix		期末考
18	Jan.12			期末考
				公佈成績

各位即將修工程數學(2B)學弟妹你們好：

暑假到了，大家對自己的時間有何安排？希望你們過的很充實，玩也玩到了，書也唸了一些！在暑假期間若能好好利用安排時間溫預複習功課，必能使你下學期的修課更加順利，不要到期末成績公佈後再勞煩老師。

工程數學與學習河海工程的我們有密不可分的關係。不管是現在，還是在往後的學習、工作、研究或實際工程問題，提供想像與思考的空間，並和物理觀念一氣呵成；即能掌握理論之精義，並實際感受數學模式對物理現象的詮釋，所以工數可說是一門活的學問。然而在學習工數的過程中，微積分係其前置基礎。所以工欲善其事必先利其器，如果把基礎打好，做事才會事半功倍。微積分要好不二法門就是多演算、多練習、多思考。微積分不好想必修起工數來會相當吃力。由微積分授課教師(許玉平 博士)處得知大家學習微積分效果不盡理想，甚至有 1/3 同學相當吃力。且由於時間關係，微積分沒教到的部份，包括：

1. Infinite Series (including power series and Taylor series)
2. Double integrals in polar coordinates
3. Triple integrals (including cylindrical and spherical coordinates)
4. Vector Calculus (including line and surface integrals)

所以大家在暑假這段期間，請不要虛擲光陰，一定要好好下苦工把這個洞補起來。身為學長的我們以前沒機會收到此信，希望能與大家共勉之。為了方便下學期我們服務團隊(助教與小老師群)了解學生程度，我們整理了一些微積分基本題型，希望大家能確實練習。請大家做完後傳回來，我們會幫忙批閱，希望這不會對您們佔用太多時間，並且協助你們的學習。希望我們的用心可以得到學弟妹善意的回應。Welcome to visit our web site for more detail. http://msvlab.hre.ntou.edu.tw/

河工系學長們給學弟的一句話：

博士級	碩士級	大學級
博士後學長 李應德： 學海無涯，唯勤是岸。 博士班學長 李家瑋： 堅定的意志力，會使你成功。	碩二學長 李文哲： NO pains, NO gains. 碩二學長 徐胤祥： Be studious in your profession, and you will be learned. 碩一學長 范羽： 做學問無他，唯有放心而已。 碩一學長 陳逸維： 一分耕耘一分收穫。 碩一學長 郭柏伸： 學弟妹要珍惜老師與學長給的資源。	大四學長 陳衍穎： 凡事不可操之過急，欲速則不達。 大四學長 江立傑： 世界上最貴的莫過於時間，精算且利用每分每秒。 大三學長 蔡育仁： 學弟妹們，多加利用暑假不用修課的時間溫習功課。 大三學長 張峻閔： 莫貪一時樂，種下未來苦，讀書的目的是再管理自己，學問的境界在服務他人，機會總是留給主動積極的人。 大二學長 賴偉文： 成功是留給準備好的人。

敬上 July.2011

cc：河工系學長們 給學弟妹的一封信

1. $\frac{d}{dx}(\cos x + e^x) = ?$

2. $f(x,y) = e^y \sin(x)$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

3. Cosh(x) = ? (hyperbolic cosine)

4. $g(y) = \int_0^{2\pi} f(x,y)dx$ what is the dummy variable x or y ?

5. $\int_0^1 \int_0^y 1 dxdy = \int_0^1 \int_?^? dydx = ?$

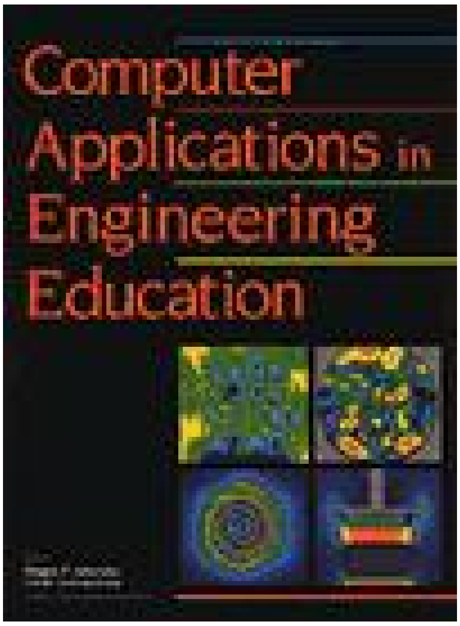
6. If $y = \sqrt{1-x^2}$, find the arc length from (0,1) to(1,0).

時光匆匆，轉眼間也從海洋大學畢業了，這所孕育無數人材的母校，回首過去，大學生活快樂充實，充滿笑與淚，盡是對基隆這座城市滿滿的不捨與感慨。

大學生所享受的自由，是多麼的無拘無束。那段青澀的日子裡，工學院外的海堤，堤外的大海伴我過無數的日子，大學就像一座寶山，需要努力的向裡挖掘，才能找到屬於自己的寶藏。個人有幸能夠遇到許許多多令人感恩的啟蒙老師，也開始了我的尋寶之旅，何其有幸進入陳正宗特聘教授所率領的力學聲響振動實驗室，在那我不僅得以在陳老師指導下，也可與一些研究所學長切磋學習，在在讓我學到更多更廣的知識，才發現學海無涯的真正意義。

還記得陳老師常說的兩句話：「*作學問、趣味趣味*」、「*作研究就是埋頭苦幹，終有一天會闖出自己的一條路*」，這些話令學生覺得要達到自我的目標，就是必須不斷的投入與努力，在陳教授與機械系劉進賢教授共同指導下參與 95 年度國科會大專生研究計畫，也相當幸運的此份成果也受到中國工程師學會的青睞，學生論文競賽獲得海大中工會特優、全國中工會優等，投稿於國外期刊也相當順利，大學時期便能有一篇SCI期刊論文發表，相信這是我在海大四年的一個記錄與美好的回憶。當得知文章能夠被接受時，學生真是受寵若驚，即使在台清交成等所謂名校，大學生能有如此小小成就，也相當少見，各位學弟妹請勿妄自菲薄，人材也是需要環境的磨練，才能發光發熱。

回想一路走來，求學過程的順利，學生相當感激所遇到的海大各個師長，以及呂學育老師、劉進賢教授、陳正宗終身特聘教授。海洋大學開創了我的視野，也期勉目前就讀於海洋大學的學弟妹們，能夠善用海洋大學的資源，開創自己美好的未來。



CAEE 期刊 (SCI)

作者於 NTOU/MSV Lab 留影

李家瑋 同學攝影

校友來鴻

海洋大學開創我的視野

蔡明宏　（河工系96級校友）

時光匆匆，轉眼間也從海洋大學畢業了，這所孕育無數人材的母校，回首過去，大學生活快樂充實，充滿笑與淚，盡是對基隆這座城市滿滿的不捨與感慨。

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海大河工三 B　李家瑋　 小老師心得

在因緣際會下，很榮幸能得到資工系李孟書老師推薦，擔任９５學年度第二學期「積極性補強教學」的課輔助教，我所輔導的班級是機械 1B 的微積分。今年更獲河工系陳正宗終身特聘教授的厚愛接任河工 2B 的工數小老師，跟各位一樣在這一學期要扮演好小老師該有的角色與完成任務。

在上學期剛開始上課的時候，我曾想過，因爲我本身是河工系的同學，會不會因爲所輔導的班級是機械 1B，平時的互動本來就比較少，而感覺怪怪的。但我想這次機會不但可以磨練自己表達能力，以及幫助需要補強的同學，而自己以前在李孟書老師所教的微積分，學習的頗有心得，況且平常班上同學常與我討論、一起學習，因此就接受了這個任務。

在上學期接這工作時，起初時當然會很緊張，因爲我以前都沒有任何教學經驗，甚至連家教都沒有教過，頂多只是教有問題來問我的同學，不知道該以怎樣的形式來教導同學。而老師就建議我以過來人的經驗，來教導學弟，分享自己在學習微積分的經驗，以及解題時的關鍵或是學習所遇到的問題爲何，亦即將自己怎樣走過來的經歷傳承吧。

而在上學期的第一堂課，我最擔心的事情果真發生了，只有一位同學來上課，讓我感受到有點沮喪，也能體會到同學不來上課時，認真盡職老師的感受及心情。而後來老師就規定小考不好的同學一定要來上課，也因爲這樣，後來上課的同學也就比較多了，這樣上起課來也讓自己比較有熱忱。一學期下來，也對同學比較有認識。而到了下學期的補強教學，每次都有不少同學來上課，有一學期的經驗了，就比較不會緊張，也能比較清楚該如何表達同學們才聽的懂。

而在準備微積分課程的方面，我發現到大部分的同學的問題都是一致的，都是不知道公式怎樣運用，題目怎麼解，然而這一切問題都是缺乏練習所造成的。因此我在講解算例時，都會以我以前是怎麼解決，以及可能會遇到的問題來告訴同學，也時常提醒同學，作業一定要自己動手寫過，這樣才會是自己的東西，補強教學，對已經會的同學只是再加深層印象，對不會的同學，便是提供一個解題方法，但要學會實際上還是要自己練習過，才會是屬於自己的東西。

經過了這一年的磨鍊，加深我對微積分的認識，甚至以前不太了解的地方，也因爲要準備課程的關係，而去透徹地了解，因此當課輔小老師，真的是一個很好的磨鍊，是值得各位全心全力投入的。

不過記得，一定要先說服自己，再講給學弟聽，才不至於弄亂學弟們的思緒。

海洋大學 MATHEMATICA 使用資訊

	Input	Output
Numerical Software	number → number	3 → 3^2 → 9
Symbolic Software	Symbol → Symbol	$(a+b)^2 \rightarrow a^2+2ab^2+b^2$

Available symbolic software

- Mathematica
- Macsyma
- Maple
- Reduce
- Derive

河工三處可用

1. 河工二館　504 教室
2. 河工二館　307 會議室
3. 河工二館　306 室 力學聲響振動實驗室

歡迎多加使用

Mathematica 資訊 .doc
楊熙民製

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第一章 一階常微分方程式

為何學工數?

海大河海系 陳正宗

- 數學、力學與實驗三位一体, 缺一不可。
  There is nothing more practical than the right theory; however,
  whether the theory is right or not depends on the experiment.
  從事理論分析的人, 只有自己相信結果是對的。
  從事實驗分析的人, 則是別人相信結果是對的。
- 工數可提供思考的工具, 擴大想像空間, 若能配合物理思考模式, 將可收事半功倍之效。
- 木蘭詩
  雄兎脚蹼朔, 雌兎眼迷離, (output, phenomenon)
  兩兎傍地走,(input)
  焉能辨我是雌雄？(system identification)
  input → black box → output
  若不提耳朶？(real input)
  厚臉皮？老化？
- 要學習一門工程技術, 如果沒有一點工數的基礎, 幾乎一本書也看不懂。
- 回家種田最有用。
- 研究所進階之路。
- 學好工數, 有助學習工程科學。
- 工數是造物者透露天機的一個管道。

- Function of space :

  String subjected to external force

  $$y"(x) = p(x)$$

- Function of time :

  Fall of free body

  $$m\ddot{x}(t) = mg$$

- Function of space and time

  Diplacement response of tall building

  $$\rho\ddot{u}(x,t) = -EIy''''(x,t)$$

  Wave phenoomenon

  $$\rho\ddot{u}(x,t) = EAu''(x,t)$$

- 宇: 上下四方
- 宙: 古往今來
- 一維: rod
- 二維: plate
- 三維: solid(dam)
- 單變數: x or t (ODE)
- 多變數: **x** and t (PDE)

函數

海大河海系 陳正宗

independent variable $\mathbf{x}, t$ (master set)

dependent variable $f(\mathbf{x}, t)$ (slavery set)

function of time $f(t)$

function of space $f(\mathbf{x})$, 1-D, 2-D, 3-D

function of space and time $f(\mathbf{x}, t)$

unit step function

jump function, $tan^{-1}x$, Heaviside function

impulse function

Dirac-delta function

symmetric and antisymmetry

function decomposition, unique, loading pattern

trivial function

even or odd

filter function

inverse function, one to one

real or complex

function generator

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- 自由落体 (無阻力)

$$m\ddot{x}(t) = mg$$

- 終端速度 (有阻力)

$$m\dot{v}(t) = mg - bv^2$$

- 單擺周期效應

$$ml^2\ddot{\theta}(t) + mglsin(\theta) = 0$$

- 自由振動

$$m\ddot{x}(t) + kx(t) = 0$$

- 電路振盪

$$L\ddot{I}(t) + R\dot{I}(t) + \frac{1}{C}I(t) = \dot{E}$$

- 洩水問題

$$\dot{h}(t) = -k\sqrt{h}$$

- 吊橋繩索問題

$$y\text{''}(x) = k\sqrt{1 + (y'(x))^2}$$

- 半衰期

$$\dot{y} = -ay(t)$$

- mixing of solution

$$\frac{dQ}{dt} = 10 - \frac{Q}{20}, Q(0) = 100$$

If $Q(t_f) = 150$, find t_f. (nonhomogeneous standard first order ODE)

- leakage through an orifice

$$\pi(2yR - y^2)dy = -\pi r^2 \sqrt{2gy}\, dt, y(0) = R$$

If $y(t_f) = 0$, find t_f.(separable)

- dissolving of a solid in a liquid

$$\frac{dQ}{dt} = \frac{k}{120}(20 - Q)(40 - Q), Q(0) = 0$$

If $Q(12) = 4$, find $k = \frac{1}{2} ln(9/8)$. (separable, fraction type)

- heat loss from a pipe

$$dT = \frac{Q}{2\pi k}\frac{dr}{r}, T(r_0) = T_0$$

(separable)

- laminar fluid flow

$$\frac{d}{dr}(r\frac{dv}{dr}) = \frac{-r(p_0 - p_1)}{nl}, v(0) = finite, v(a) = 0$$

(separable)

- orbital motion

$$\frac{dv}{dr} - \frac{2}{r}v = \frac{r}{r_c}(r_c - r)v^{-1}$$

(Bernoulli form)

- exponetial population growth

$$\frac{dN}{dt} = \frac{r}{100}N, N(0) = N_0$$

(separable)

- A finite population model

$$\frac{dN}{dt} = (a - bN)N, N(0) = N_0$$

(separable)

Method I:

Governing equation:

$$m\ddot{x}(t) + kx(t) = 0$$

subjected to

$$x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

Physical point: equlibrium of Newton's law
Mathematical point: second order linear ODE subjected to two initial conditions

Method II:

Governing equation:

$$\frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}kx^2(t) = \frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}kx_0^2$$

subjected to

$$x(0) = x_0$$

Physical point: Conservation of mechanical eneregy (strain enegy and kinetic energy)
Mathematical point: first order linear ODE subjected to one initial condition

Method III:

In senior high school, we used projection concept to understand the motion of SHM.

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Differential equation

The equation which contains the differential operator with respect to unknown function is called differential equation.

Mass inertia

$$m\ddot{y}(t) = mg$$

Difference equation

The equation which contains the difference operator with respect to unknown sequence is called difference equation.

$$y_{n+1} - y_n = 0, n = 0, 1, 2, 3, \cdots$$

Integral equation

The equation which contains the integral operator with respect to unknown function is called integral equation.

Material inertia

$$\int_0^t (t-\tau)y(\tau)d\tau = p(t)$$

Integral-differential equation

The equation which contains the integral and differential operators with respect to unknown function is called integral-differential equation.

Mass and material inertia

$$m\ddot{y}(t) + \int_0^t (t-\tau)y(\tau)d\tau = p(t)$$

ODE, PDE

Order

$$\dot{y} = \alpha y \ (first\ order)$$

$$\ddot{r} = -gR^2/r^2 \ (second\ order)$$

$$\ddot{x}(t) - 3\dot{x}(t) + x(t) = cos(t), \ (second\ order)$$

$$[y''''x]^{5/2} - 2y'' = cos(x), \ (Fourth\ order)$$

$$F(x, y, y', ..., y^n(x)) = 0 \ (nth\ order)$$

Linearity(linear or nonlinear)

$$y_1, y_2\ satisfy\ homo.\ linear\ ODE \rightarrow y_1 + y_2\ satisfy\ homo.\ linear\ ODE$$

Homogeneity(homogeneous or nonhomogeneous)

Problem statement

$$\frac{dy}{dt} = \alpha y, y(0) = y_0$$

Method 1: by view

$$y(t) = y_0 e^{\alpha t}$$

Method 2:separation of variables

$$\frac{dy}{dt} = \alpha y, y(0) = y_0$$

$$\int \frac{dy}{y} = \int \alpha dt$$

$$ln(y) = \alpha t + c$$

$$y(t) = e^{\alpha t + c} = Ke^{\alpha t}$$

$$y(0) = y_0 \;\rightarrow\; y(t) = e^{\alpha t + c} = y_0 e^{\alpha t}$$

Method 3: series solution

$$y(t) = y_0 + y_1 t + y_2 t^2 + \cdots + y_n t^n + \cdots$$

$$\dot{y}(t) = y_1 + 2y_2 t + \cdots + n y_n t^{n-1} + \cdots$$

$$\alpha y(t) = \alpha y_0 + \alpha y_1 t + \cdots + \alpha y_n t^n + \cdots$$

Comparing the coefficients, we have

$$y_n = \frac{1}{n!}\alpha^n y_0$$

$$y(t) = y_0 e^{\alpha t}$$

Method 4: succesive iteration method

$$\int_0^t dy(t) = \alpha \int_0^t y(t)dt \;\rightarrow y(t) = y(0) + \alpha \int_0^t y(t)dt$$

$$y_1(t) = y_0$$

$$y_2(t) = y_0 + \alpha y_0 t$$

$$\cdots = \cdots$$

$$y_n(t) = y_0 + y_0 \alpha t + \frac{1}{2!} y_0 (\alpha t)^2 + \frac{1}{3!} y_0 (\alpha t)^3 + \cdots$$

$$y(t) = y_0 e^{\alpha t}$$

疊代法

點疊代 $x_0, x_1, x_2 \cdots$

$$x_{n+1} = g(x_n)$$

收斂準則: $\mid g'(x_0) \mid < 1$, 亦即

$$(1 - f'(x_0)) < 1, \rightarrow f'(x_0) > 0$$

Examples: $x^2 - 3 = 0$

$x_0 = 2.0$ initial guess:

Case 1: $x = \frac{1}{2}(x + \frac{3}{x})$ (OK)

Case 2: $x = x^2 + x - 3$ (NG)

Case 3: $x = \frac{3}{x}$ (NG)

函數疊代 $x_0(t), x_1(t), x_2(t), \cdots$

$$x_{n+1}(t) = \mathcal{F}\{x_n(t)\}$$

如

$$\dot{y}(t) = ay(t)$$

疊代式

$$y(t) = y(0) + \int_0^t ay(\tau)d\tau$$

土壤阻尼動力方程:

$$-kx(t) = m\ddot{x}(t) - \frac{k\eta}{\pi}\int_{-\infty}^{\infty}\frac{x(u)}{(t-u)}du$$

向量疊代 $\tilde{x}_0, \tilde{x}_1(t), \tilde{x}_2(t), \cdots$

$$\tilde{x}_{n+1}(t) = \mathcal{F}\{\tilde{x}_n(t)\}$$

如

$$\frac{d\tilde{x}(t)}{dt} = \mathbf{A}\tilde{x}(t)$$

疊代式

$$\tilde{x}(t) = \tilde{x}(0) + \int_0^t \mathbf{A}\tilde{x}(\tau)d\tau$$

State: point, function and vector.

Table 1: Successive iteration method for nolinear algebraic equation, ordinary differential equation ,and vector equation

Question	$f(x)$	$\dot{y}(t) = ay(t)$	$\dot{\tilde{x}}(t) = A\tilde{x}(t)$
Operaton	*algebraic*	*integration*	*integration*
Modified form	$x_{n+1} = g_n(x)$	$y_{n+1}(t) = \mathcal{F}\{y_n(t)\}$	$\tilde{x}_{n+1} = \mathcal{F}\{\tilde{x_n}\}$
Convergence	$\mid g^{'}(\alpha) \mid < 1$	?	?
State	*point*	*function*	*vector*

where $\mathcal{F}\{y(t)\}$ *denotes functional of* $y(t)$

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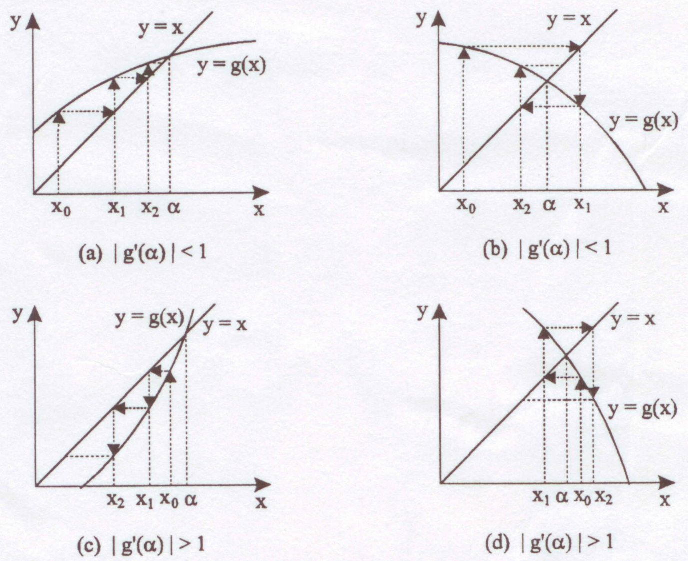


圖一 直接代入法

$$\dot{P}(t) = P(t)(\beta - \delta P) \tag{1}$$

Three cases of initial conditions:
Case 1: unreasonable

$$P(0) < 0$$

Case 2: grow to be saturated

$$0 < P(0) < \frac{\beta}{\delta}$$

Case 3: decay to be saturated

$$P(0) > \frac{\beta}{\delta}$$

general solution is :

$$P(t) = \frac{\beta}{\delta + [\frac{\beta}{P(0)} - \delta]e^{-\beta t}}$$

Asymptotic population= $\frac{\beta}{\delta}$.

$$dy/dx = xy, y(1) = 2 \tag{2}$$

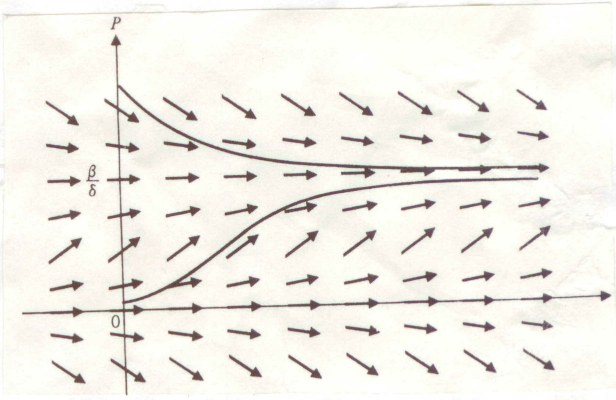
$$dy/dx = x/y, y(1) = 0 \tag{3}$$

$$dy/dx = x^2 + y^2, y(0) = 1 \tag{4}$$

方向場作圖求解常微分方程

海大河海系　　陳正宗

$$\dot{P}(t) = P(t)(\beta - \delta P) \tag{1}$$



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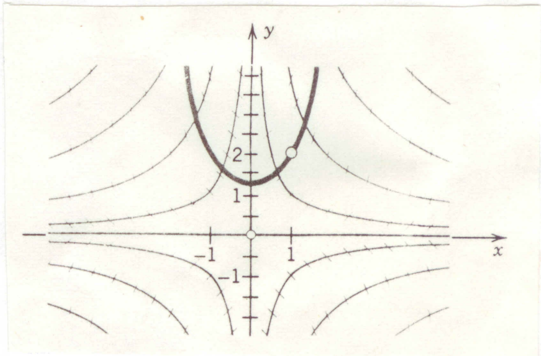

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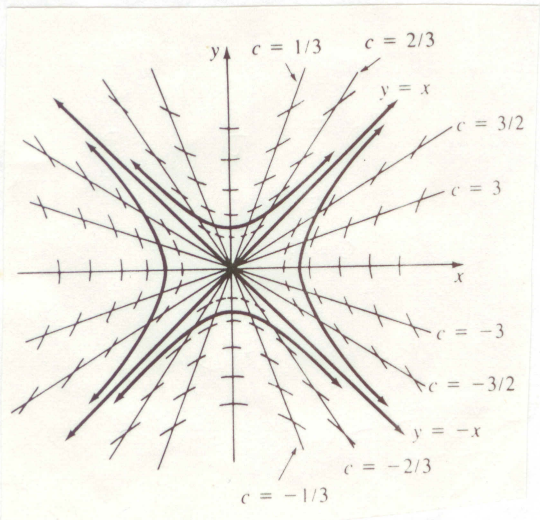
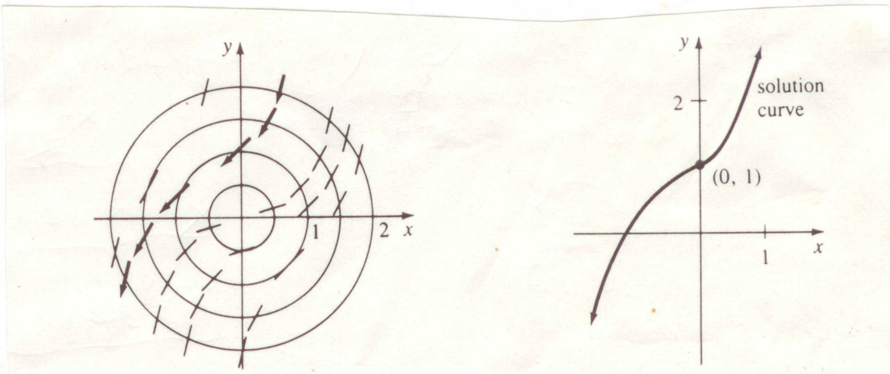


$$dy/dx = xy, y(1) = 2 \tag{2}$$

$$dy/dx = x/y, y(1) = 0 \tag{3}$$

$$dy/dx = x^2 + y^2, y(0) = 1 \tag{4}$$



海洋大學河海工程學系2007 工程數學（一）2B 班; 方向作圖法

1. The four ODEs are defined as follows:

$$y' = y^{3/5} \tag{1}$$

$$y' = (y^2 - 4)(y - 4) \tag{2}$$

$$y' = \frac{2xy}{1 + y^2} \tag{3}$$

$$y' = 1 + x + y \tag{4}$$

$$y' = y(1 - y) \tag{5}$$

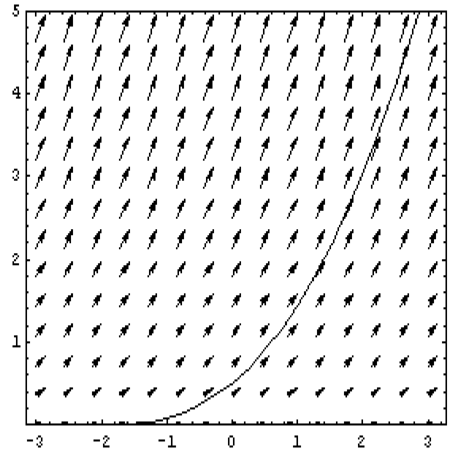
$$y' = xy \tag{6}$$

The solutions are plotted in the following figures. Write down the corresponding characters (a), (b), (c) and (d) in the table.(100%)

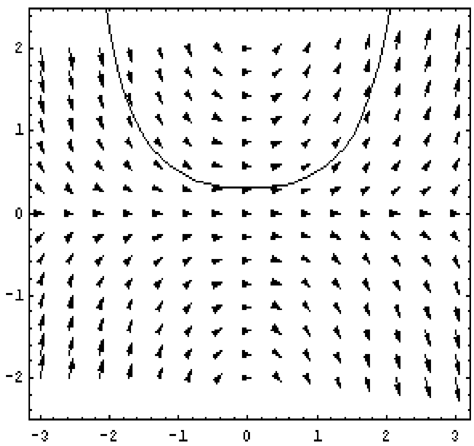
Table 1: Solutions mapping to each ODE

ODE	Eq.(1)	Eq.(2)	Eq.(3)	Eq.(4)	Eq.(5)	Eq.(6)
Fig.	a	b	c	d	h	f

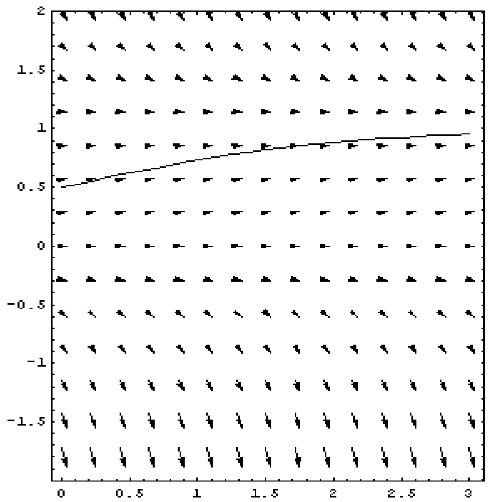
$y'[x] == y[x]^{3/5}$



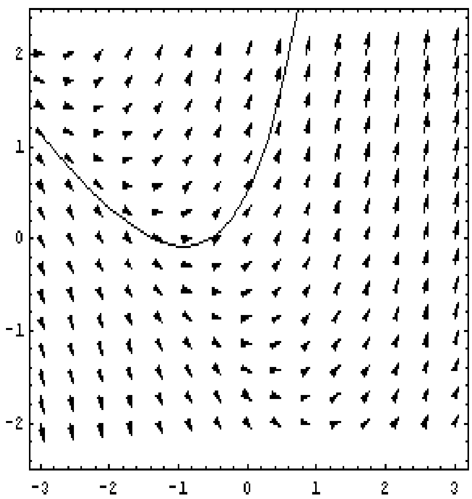
$y'[x] == x * y[x]$



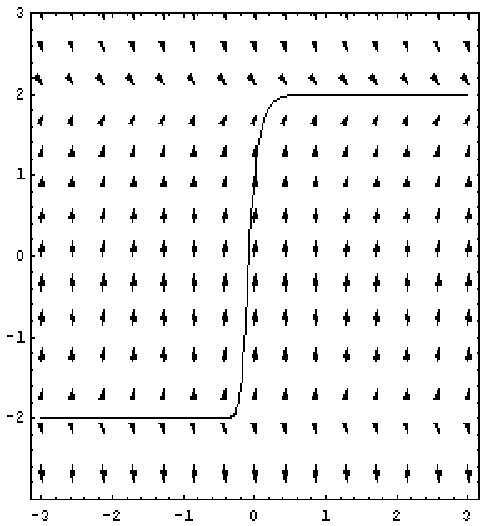
$y'[x] == y[x] * (1 - y[x])$



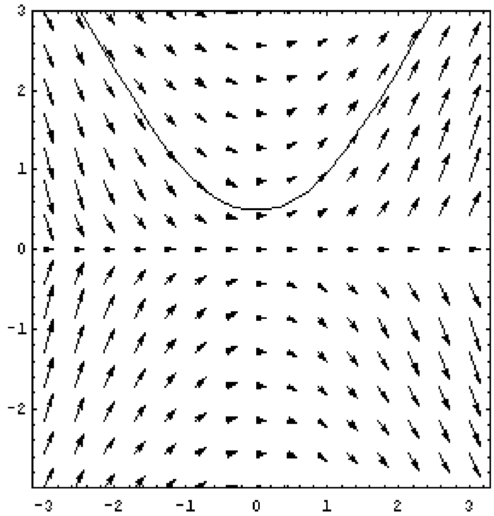
$y'[x] == 1 + x + y[x]$



$y'[x] == (y[x]^2 - 4) * (y[x] - 4)$



$y'[x] == \frac{2 * x * y[x]}{1 + y[x]^2}$



SCC Homepage | Calculus I | Calculus II | Calculus III | Diff Eq | Calculus Homepage

Notes For Exam 1
Notes For Exam 2
Notes For Exam 3
Notes For Exam 4
Differential Equations Links

Syllabus
Homework
Textbook
Angel
Maple Applications Student Help Center

DIFFERENTIAL EQUATIONS

Maple will be used extensively in this course and will be available in class and during exams. The student version is available for purchase at a reduced rate. Click here for purchase information. It will not be required that you purchase your own copy of Maple but highly recommended since it is a wonderful tool and very helpful in this course and Calculus III.

National Hurricane Center FEMA News Release 2004--Hurricane Planning in Louisiana

December 6 Is Exam Re-Take Day

August 23 Sections 2.2, 2.5 Notes For Exam 1	August 30 Sections 2.4, 2.3 Notes For Exam 1	September 6 Section 2.6 Notes For Exam 1	September 13 Section 3.1 Notes For Exam 1	September 20 Exam I Exam 1 Take-Home Problems
September 27 Sections 4.1, 4.3 Notes For Exam 2	October 4 Section 4.5 Notes For Exam 2	October 11 Sections 4.5, 5.1 Notes For Exam 2	October 18 Exam II Exam 2 Take-Home Problems	October 25 Sections 7.1-7.3 Notes For Exam 3
November 1 Sections 4.6, 4.7, 7.1-7.3 Notes For Exam 3	November 8 Exam III No Exam III Take-Home	November 15 Section 6.1 Notes For Exam 4	November 29 Sections 6.1, 6.2 Exam 4 Take-Home Problems	December 6 Exam IV Exam 4 Take-Home Problems

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海大河海系 陳正宗

Separation of space and time

$$f(x,y)=g(x)h(y)\Longleftrightarrow ff_{xy}=f_xf_y$$

Proof: $\Longrightarrow$

$$ff_{xy}=ghg_xh_y$$

$$f_xf_y=g_xhgh_y$$

Proof: $\Longleftarrow$

For nontrivial case $f(x,y)\neq 0$,

$$ff_{xy}-f_xf_y=0\rightarrow\frac{ff_{xy}-f_xf_y}{f^2(x,y)}=0=\frac{\partial}{\partial y}\{\frac{f_x}{f}\}$$

$$\{\frac{f_x}{f}\}=\alpha(x)$$

$$\frac{\partial}{\partial x}(ln(f(x,y))=\alpha(x)$$

$$ln(f(x,y))=\int\alpha(x)dx+\beta(y)$$

$$f(x,y)=exp^{\phi(x)}+exp^{\beta(y)}$$

$$f(x,y)=\Phi(x)\Psi(y)$$

Example: separable

$$f(x,y)=2x^2+y-x^2y+xy-2x-2=(-x^2+x+1)(y-2)$$

$$f_x=4x-2xy+y-2$$

$$f_y=1-x^2+x$$

$$f_{xy}=-2x+1$$

$$ff_{xy}=f_xf_y$$

Example: not separable

$$f(x,y)=1+xy$$

$$f_x=y$$

$$f_y=x$$

$$f_{xy}=1$$

$$ff_{xy}\neq f_xf_y$$

Geometry meaning : surface description

$$z = g(x, y)$$

Contour line : z =constant

$$c = g(x, y)$$

Steep descent or no descent :

$$\frac{\Delta z}{\Delta \epsilon} = 0 = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \nabla g \cdot (dx, dy)$$

Corresponding ODE :

$$\frac{dy}{dx} = \frac{-\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$$

Corresponding ODE :

$$Mdx + Ndy = 0$$

Corresponding pair :

$$M = \frac{\partial g}{\partial x}, N = \frac{\partial g}{\partial y}$$

Test criteria :

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Examples :

$$(1 - sin(x)tan(y))dx + (cos(x)sec^2(y))dy = 0$$

$$M = 1 - sin(x)tan(y), N = cos(x)sec^2(y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Find $g(x, y)$:

$$\frac{\partial g}{\partial x} = 1 - sin(x)tan(y)$$

$$g(x, y) = x + cos(x)tan(y) + P(y)$$

$$\frac{\partial g}{\partial y} = cos(x)sec^2(y) + P'(y) = cos(x)sec^2(y)$$

$$P(y) = constant$$

$$g(x, y) = x + cos(x)tan(y) + c$$

where c is determined by initial or boundary conditions.

Exact form 的由來

(1) $z(x,y) = x^2 + y^2$

$z = c$

$dz = 0$

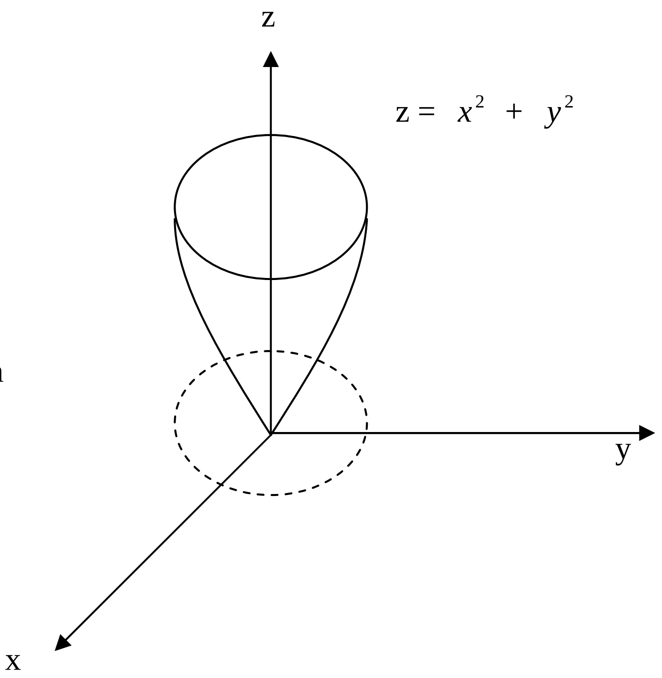
$dz = 2xdx + 2ydy = 0$

➔ $\dfrac{dy}{dx} = \dfrac{x}{-y}$

$Mdx + Ndy = 0$

$M = 2x$, $N = 2y$

Lucky $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ ➔ exact form



必可找到 z(x,y) ; such that

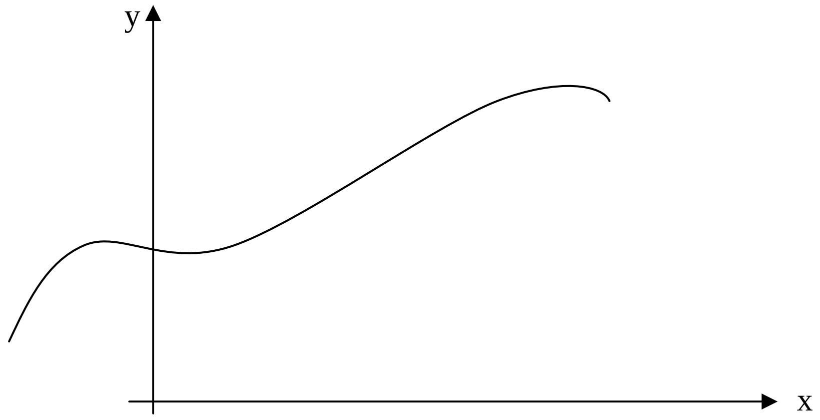
$\dfrac{\partial z}{\partial x} = M$

$\dfrac{\partial z}{\partial y} = N$

➔ $z(x,y) = x^2 + y^2$

Exact form-parameter

在 **x-y** 平面



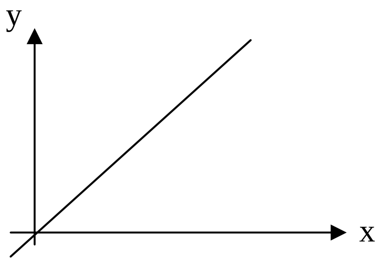
Plane curve (平面參數表示法)

[x(t) , y(t)]

$$\begin{cases} x(t)=t \\ y(t)=t \end{cases}$$ 可以得到圖形如下:

當 x=t 時，

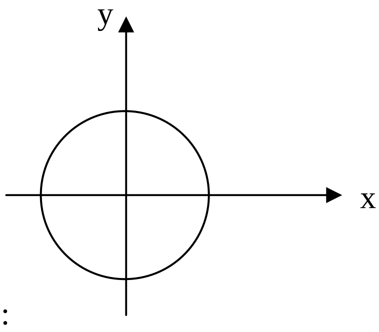
y=x 則爲一條直線



$$\begin{cases} x(t)=\cos t \\ y(t)=\sin t \end{cases}$$ 可以得到圖形如下:

可以知道 $x^2+y^2=1$

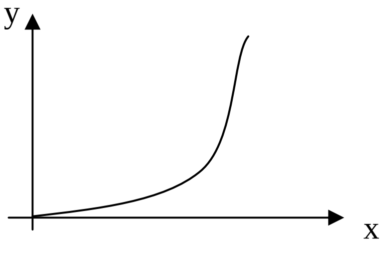
則爲一單位圓



$$\begin{cases} x(t)=t \\ y(t)=t^2 \end{cases}$$ 可以得到圖形如下:

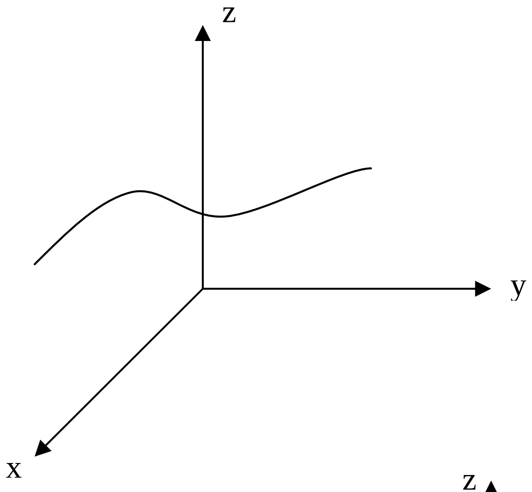
當 x=t 時，

y= x^2 則爲一條拋物線



Exact form-parameter

在 **x-y-z** 平面

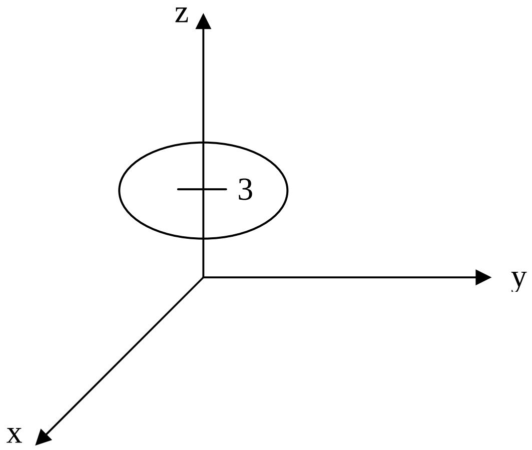


Spatial curve (空間參數表示法)

[x(t) , y(t) , z(t)]

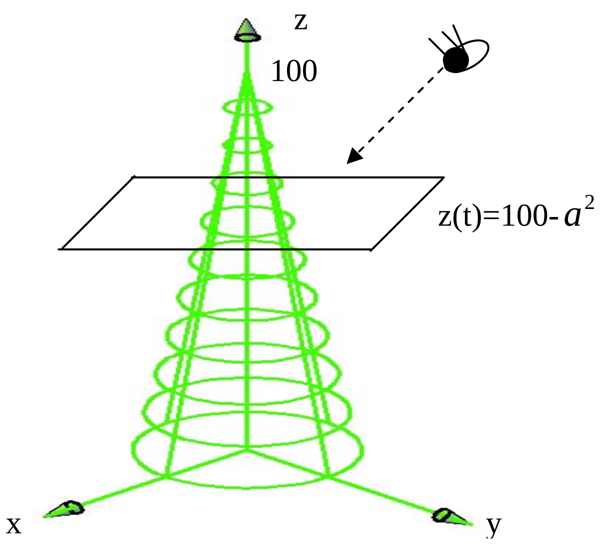
$$\begin{cases} x(t)=\cos t \\ y(t)=\sin t \\ z(t)=3 \end{cases}$$

可以得到圖形:



$z=100-x^2-y^2$ → z(x , y) , z — x, y — t

可以畫出下面的圖形:

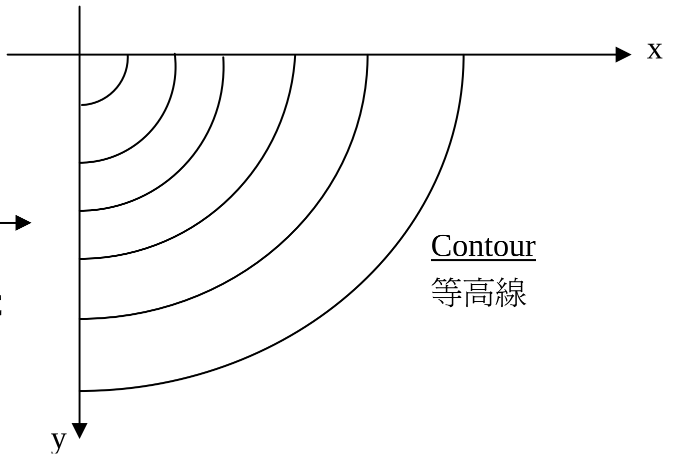


$$\begin{cases} x(t)=a\cos t \\ y(t)=a\sin t \\ z(t)=100-a^2 \end{cases}$$

在 **z=**$100-a^2$ 上有一個平面經過此圖形,切出一個半徑爲 **a** 的圓。

Exact form-parameter

ODE Solution z(x,y)=C



ODE Solution z(x,y)=C

我們可以得到 z(t)=z[x(t),y(t)]

令 z=C

C=z[x(t),y(t)]

$$\frac{dC}{dt} = \frac{dz}{dt} = 0$$

$$C = z(t) = 100 - x^2(t) - y^2(t)$$

$$\frac{dC}{dt} = 0 = -2x(t)\frac{dx(t)}{dt} - 2y\frac{dy(t)}{dt}$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$xdx + ydy = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

⟺

$$\frac{dz(t)}{dt} = 0 = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$\frac{dy}{dx} = \frac{-(\frac{\partial z}{\partial x})}{(\frac{\partial z}{\partial y})}$$

If $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$

則判別式為➔ $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$

海洋大學河海工程學系2007 工程數學（一）二B 班積分因子

1. Solve the following ordinary differential equations.

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \tag{1}$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \tag{2}$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} \tag{3}$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy} \tag{4}$$

Plot the solutions and indicate the orthogonal relationships.

ODE	Eq.(1)	Eq.(2)	Eq.(3)	Eq.(4)
Separable	No	No	No	No
Exact	No	No	Yes	Yes
Integrating factor	y^{-2}	x^{-2}	1	1
Homogeneous	Yes	Yes	Yes	Yes
Solution	$x^2 + y^2 = 2ky$	$x^2 + y^2 = 2kx$	$x^2 y - \frac{y^3}{3} = c$	$\frac{x^3}{3} - xy^2 = c$
Orthogonality	Eq.(2)	Eq.(1)	Eq.(4)	Eq.(3)

Figures:

Surface representation

$$z = z(x, y)$$

Contour line governed by the following ODE

$$\frac{dy}{dx} = \frac{-\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

Steep descent line governed by the following ODE

$$\frac{dy}{dx} = \frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}}$$

Ex.1 $z = z(x, y) = \frac{x^2+y^2-1}{2y}$

Contour line

$$\frac{dy}{dx} = \frac{x^2 - y^2 - 1}{-2xy}$$

$$(x-h)^2 + y^2 = h^2 - 1$$

Steep descent line

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2 - 1}$$

$$x^2 + y^2 - 1 = 2ky$$

Ex.2 $z = z(x, y) = x^2 + y^2$

Contour line

$$\frac{dy}{dx} = \frac{-x}{y}$$

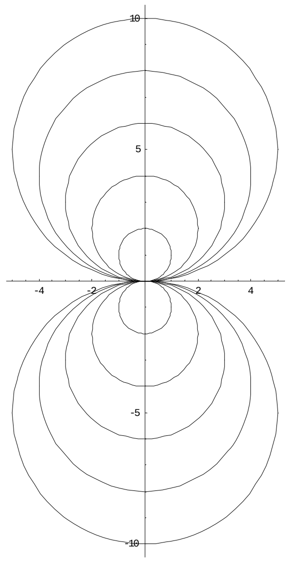
$$x^2 + y^2 = h^2$$

Steep descent line

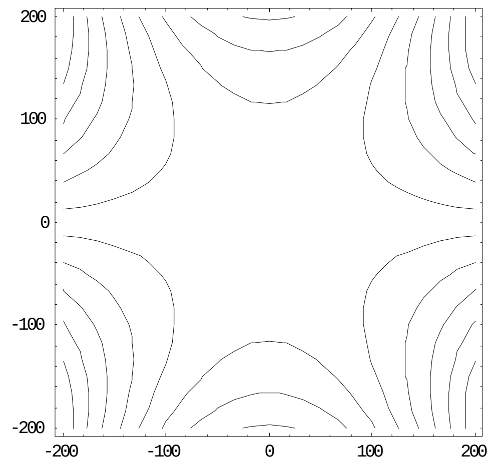
$$\frac{dy}{dx} = \frac{y}{x}$$

$$y = kx$$

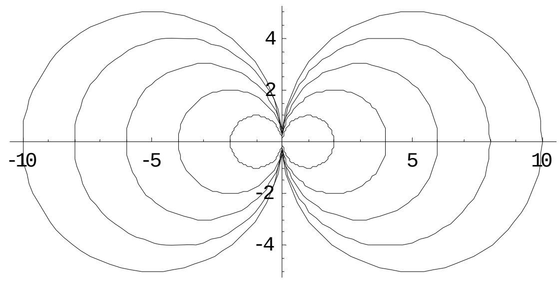
(a)



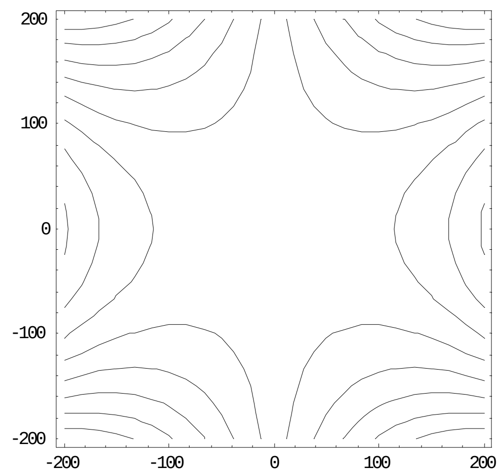
(c)



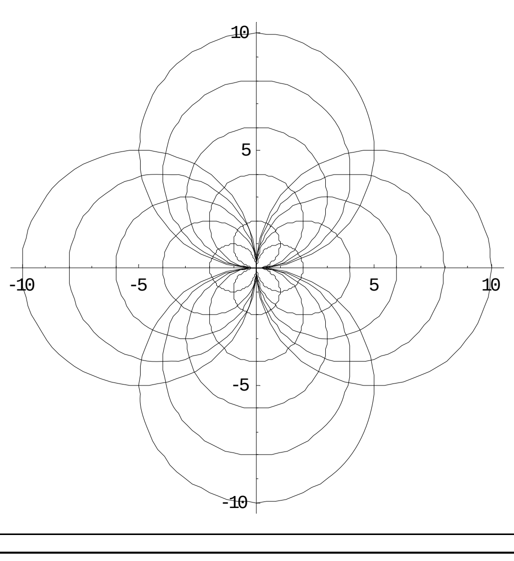
(b)



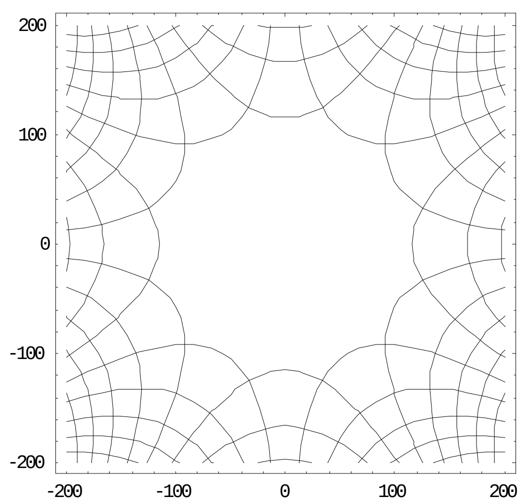
(d)



(a)+(b)

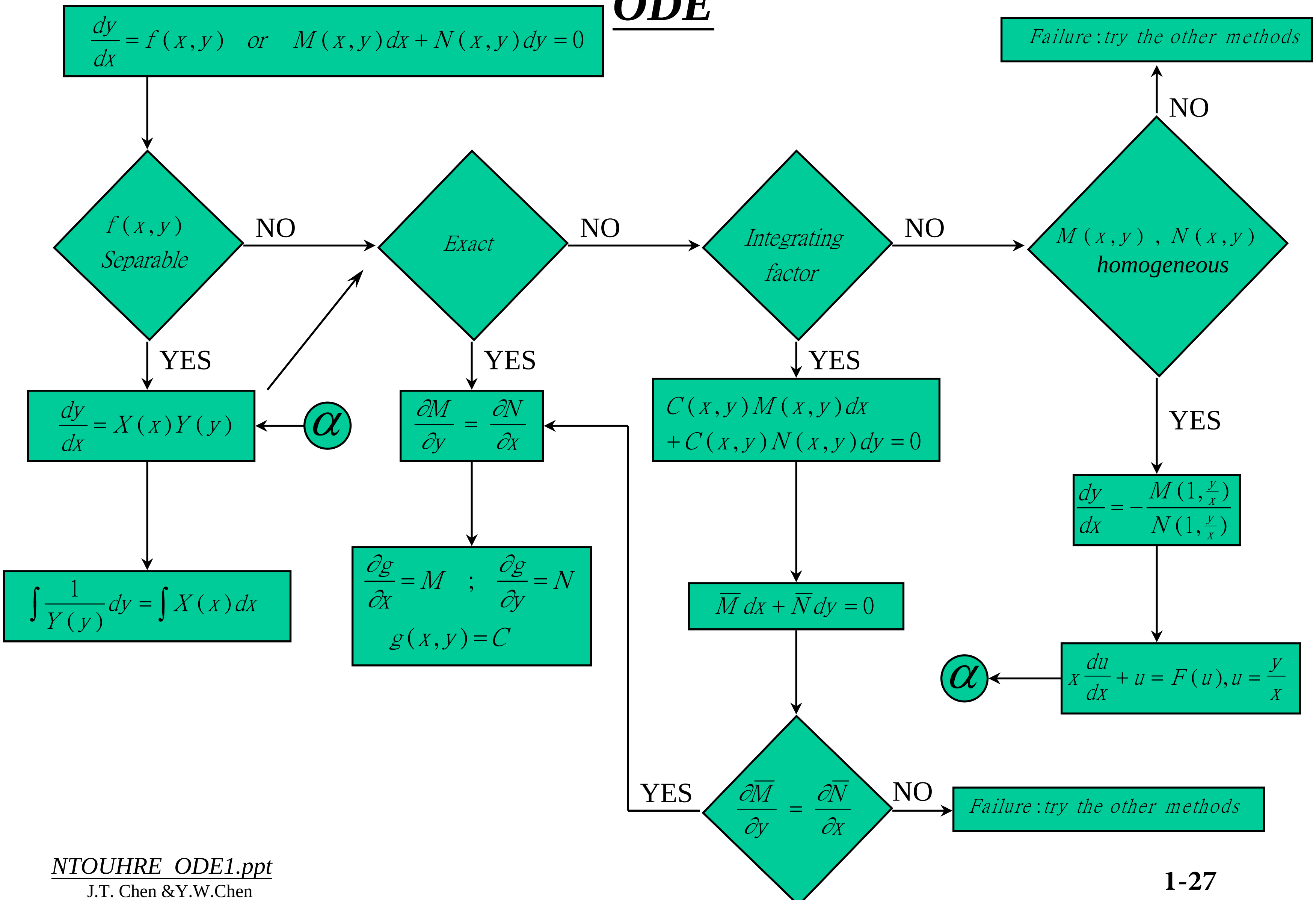


(c)+(d)



海大河工力學聲響振動實驗室　　　　河工 4A　　　　熙民製

Methods of solution for the first order ODE



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population explosion model: growth rate is proportional to current population

$$\dot{P}(t) = \alpha P(t), \alpha > 0$$

population decay model: growth rate is proportional to current population

$$\dot{P}(t) = \alpha P(t), \alpha < 0$$

population saturated model: growth rate is proportional to quadratic form of population

$$\dot{P}(t) = P(t)(\beta - \delta P)$$

Three cases of initial conditions:

Case 1: unreasonable

$$P(0) < 0$$

Case 2: grow to be saturated

$$0 < P(0) < \frac{\beta}{\delta}$$

Case 3: decay to be saturated

$$P(0) > \frac{\beta}{\delta}$$

general solution is :

$$P(t) = \frac{\beta}{\delta + [\frac{\beta}{P(0)} - \delta]e^{-\beta t}}$$

Asymptotic population= $\frac{\beta}{\delta}$.

Use Mathematica to plot the curves.

Existence and uniqueness for a solution(if f, f_y are differetial)

$$dy/dx = f(x, y), y(x_0) = y_0 \rightarrow \textit{an existent and unique solution}$$

高中(2005)	大二(ODE)2007
半衰期	$\frac{dy}{dt} = -ay \ , \ y(0) = y_0$
自由落體	$m\frac{d^2y}{dt^2} = -mg \ , \ y(0) = y_0$ $\dot{y}(0) = v_0$
簡諧運動	$m\ddot{y} + ky = 0 \ , \ y(0) = y_0$ $\dot{y}(0) = v_0$
單擺運動	$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0 \ , \ \theta(0) = \theta_0$ $\dot{\theta}(0) = \omega_0$

- Physical phenomenon : Case 1: Fall of free body
  高中物理實驗: pivoting machine, paper
  Case 2: Population dynamics:
  人口爆炸！
  Case 3: Cooling :
- Mathematical model :
  Case 1: Fall of free body

$$m\ddot{r}(t) = -GMm/r^2$$

If r is near the surface of the earth, we have

$$r(t) = R + h(t)$$

where R is the radius of the earth and h is elevation. Therefore, we have

$$\ddot{h}(t) = -GM/(R + h(t))^2$$

Since $h << R$, we can reformulate the above equation to

$$\ddot{h}(t) = \frac{-GM}{R^2}\frac{R^2}{r^2} = -g\frac{R^2}{(R+h)^2} = -g$$

Case 2: Population dynamics:

$$\dot{y}(t) = \alpha y(t), y(0) = y_0$$

Method of solution:
(1). Grossman
(2). Successive iteration method(mathematica plot)
(3). Series solution
Case 3: Newton's law of cooling:

$$\dot{y}(t) = \alpha y(t), y(0) = y_0$$

$\alpha > 0$?
$\alpha = 0$?
$\alpha < 0$?

Euler method

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

$$\frac{\Delta y}{\Delta x} = f(x_0, y_0)$$

$$
\begin{aligned}
y_1 &= f(x_0,y_0)\Delta x + y_0 && (1)\\
y_2 &= f(x_1,y_1)\Delta x + y_1 && (2)\\
y_3 &= f(x_2,y_2)\Delta x + y_2 && (3)\\
y_4 &= f(x_3,y_3)\Delta x + y_3 && (4)\\
\cdots &= \cdots && (5)\\
y_n &= f(x_{n-1},y_{n-1})\Delta x + y_{n-1} && (6)
\end{aligned}
$$

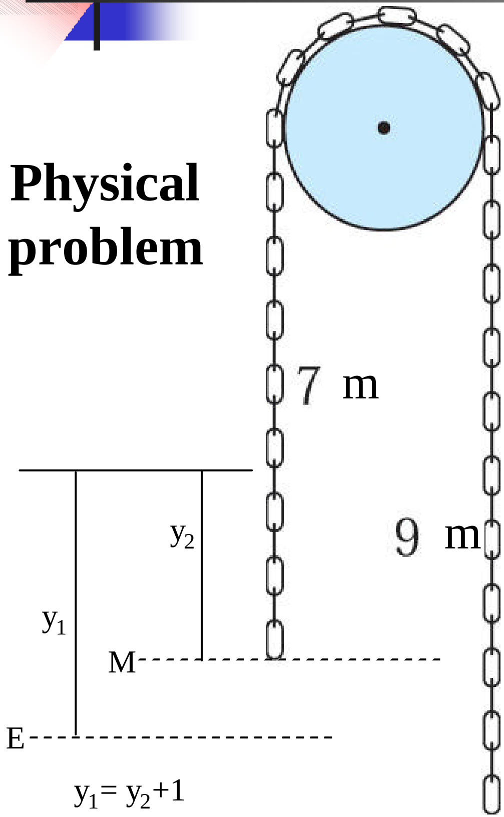
Questions:

What happens if Δx is very large ?

What happens if Δx is very small ?

Mathematical modeling

Physical problem



	y_1 system	y_2 system
	Equilibrium (E)	Max point (M)
2nd order ODE	$16\ddot{y}_1 = 2\times 9.8\times y_1$ $y_1(0) = 1$ $y_1'(0) = 0$ (exam)	$16\ddot{y}_2 = 2\times 9.8\times (y_2 + 1)$ $y_2(0) = 0$ $y_2'(0) = 0$ (HW)
1st order ODE	$v_1 \frac{dv_1}{dy_1} = 1.23 y_1$ $\frac{dy_1}{dt} = 1.11\sqrt{y_1^2 - 1}$ (exam)	$v_2 \frac{dv_2}{dy_2} = 1.23(y_2 + 1)$ $\frac{dy_1}{dt} = 1.11\sqrt{y_1^2 + 2y}$ (HW)
Comments	2nd order ODE (homogeneous ODE) (non-homogeneous BC)	2nd order ODE (non-homogeneous ODE) (homogeneous BC)
	$\int \frac{1}{\sqrt{y^2 - 1}} dy = \cosh^{-1}(y)$	v_{es} = 8.80 m/sec t_{es} = 2.5 sec

Homogeneous function with m degrees:

$$f(\lambda x_1, \lambda x_2, \cdots, \lambda x_m) = \lambda^m f(x_1, x_2, \cdots, x_m)$$

Homogeneous differential equation $f(x) = 0$:

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = f(x)$$

Nonhomogeneous differential equation $f(x) \neq 0$:

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = f(x)$$

Homogeneous function:

$$M(x,y) = M(\lambda x, \lambda y)$$

$$N(x,y) = N(\lambda x, \lambda y)$$

Transformation from nonseparable problem to separable problem:

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

$$\frac{dy}{dx} = \frac{-M(1,y/x)}{N(1,y/x)} = F(y/x)$$

Setting $y = ux$, we have

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

Then the nonexact form can be reduced to

$$xdu = (F(u) - u)dx$$

$$\frac{1}{x}dx = \frac{1}{(F(u) - u)}du$$

一階微分方程解法

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linear first order ODE

$$\dot{y}(x) + a(x)y(x) = f(x)$$

Linearity :

$$\dot{y}_1(x) + a(x)y_1(x) = 0$$
$$\dot{y}_2(x) + a(x)y_2(x) = 0$$
$$\dot{Y}(x) + a(x)Y(x) = 0$$
$$Y(x) = y_1(x) + y_2(x)$$

y_1 and y_2 are solutions $\rightarrow$ $Y(x)$ is solution.

Method 1: integration factor for $a(x) = a$ only

$$e^{ax}\dot{y}(x) + e^{ax}ay(x) = e^{ax}f(x)$$
$$\frac{d\{e^{ax}y(x)\}}{dx} = e^{ax}f(x)$$
$$e^{ax}y(x) = \int e^{ax}f(x)dx + c$$
$$y(x) = e^{-ax}\int e^{ax}f(x)dx + ce^{-ax}$$

Method 1: integration factor for $a(x)$ only

$$e^{\int a(x)dx}\dot{y}(x) + e^{\int a(x)dx}a(x)y(x) = e^{\int a(x)dx}f(x)$$
$$\frac{d\{e^{\int a(x)dx}y(x)\}}{dx} = e^{\int a(x)dx}f(x)$$
$$e^{\int a(x)dx}y(x) = \int e^{\int a(x)dx}f(x)dx + c$$
$$y(x) = e^{-\int a(x)dx}\int e^{\int a(x)dx}f(x)dx + ce^{-\int a(x)dx}$$

Method 2: variation of parameters $a(x)$ only

$$y(x) = u(x)v(x)$$
$$uv' + u'v + a(x)uv = f(x)$$
$$uv' + (u' + a(x)u)v = f(x)$$
$$uv' = f(x)$$
$$(u' + a(x)u) = 0$$
$$u(x) = c_1e^{-\int a(x)dx}$$
$$v' = u^{-1}f(x) = \frac{1}{c_1}e^{\int a(x)dx}f(x)$$
$$v(x) = \frac{1}{c_1}\int e^{\int a(x)dx}f(x)dx + c_2$$
$$y(x) = e^{-\int a(x)dx}\{\int e^{\int a(x)dx}f(x)dx\} + ce^{-\int a(x)dx}$$

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1. Transformation of dependent variable: nonseparable to separable

   Transformation from nonseparable problem to separable problem:

   If $M(x,y)$ and $N(x,y)$ are homogeneous with the same degrees, we have

   $$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

   $$\frac{dy}{dx} = \frac{-M(1,y/x)}{N(1,y/x)} = F(y/x)$$

   Setting $y = ux$, we have

   $$\frac{dy}{dx} = u + x\frac{du}{dx}$$

   Then the nonexact form can be reduced to

   $$xdu = (F(u) - u)dx$$

   $$\frac{1}{x}dx = \frac{1}{(F(u) - u)}du$$

   $(x, y(x)) \rightarrow (x, u(x))$ where $u(x) = y(x)/x$.

2. Transformation of independent variable(Cauchy-Euler equation): variable coef. ODE to const. coef. ODE

   Variable coefficient ODE :

   $$t^2\ddot{y}(t) + at\dot{y}(t) + by(t) = 0$$

   Change of independent variable:

   $$t = e^x, x = ln(t)$$

   Constant coefficient ODE :

   $$Y"(x) + (a-1)Y'(x) + bY(x) = 0$$

   $(t, y(t)) \rightarrow (x, Y(x))$ where $Y(x) = y(t)$.

求解微分方程的轉換方法

海大河海系 陳正宗

1. Transformation of dependent variable(Bernoulli equation) : nonlinear form to linear

   $$\dot{y}(x)+a(x)y(x)=f(x)y^n(x)$$

   Change of dependent variables: nonlinear transformation

   $$z(x)=y^{1-n}(x)$$

   Linear ODE : standard form

   $$\dot{z}(x)+(1-n)a(x)z(x)=(1-n)f(x)$$

   Special case : nonlinear form

   $$\dot{y}(x)+a(x)y(x)=f(x)\,y\,ln(y)$$

   Change of independent variables: nonlinear transformation

   $$z(x)=ln(y)$$

   Linear ODE : standard form

   $$\dot{z}(x)+a(x)=f(x)z$$

2. Transformation of independent variable(Cauchy-Euler equation): variable coef. ODE to const. coef. ODE

   Variable coefficient ODE :

   $$t^2\ddot{y}(t)+at\dot{y}(t)+by(t)=0$$

   Change of independent variable:

   $$t=e^x, x=ln(t)$$

   Constant coefficient ODE :

   $$Y''(x)+(a-1)Y'(x)+bY(x)=0$$

3. Integral transformation of independent variable(Cauchy-Euler equation): diff. operator to algebraic operator

   $$F(s)=\int_0^\infty y(t)e^{-st}dt$$

Bernoulli 方程與人口成長模式

海大河海系 陳正宗

1. Transformation of dependent variable(Bernoulli equation) : nonlinear form to linear

   $\dot{y}(x) + a(x)y(x) = f(x)y^n(x)$

   Change of dependent variables: nonlinear transformation

   $z(x) = y^{1-n}(x)$

   Linear ODE : standard form

   $\dot{z}(x) + (1-n)a(x)z(x) = (1-n)f(x)$

   Special case : nonlinear form

   $\dot{y}(x) + a(x)y(x) = f(x)\, y\, ln(y)$

   Change of independent variables: nonlinear transformation

   $z(x) = ln(y)$

   Linear ODE : standard form

   $\dot{z}(x) + a(x) = f(x)z$

2. Example: logistic population model

   Bernoulli ODE :

   $y' - Ay = By^2$

   Separable ODE :

   $y' = Ay - By^2 = (y)(A - By)$

   Change of independent variable:

   $u = 1/y$

   ODE for u :

   $u' + Au = B$

   Solution:

   $$u(x) = \frac{1}{(B/A) + ce^{-Ax}}$$

數學模式-解的存在、唯一與合理性？

海大河工系 陳正宗

- Existence :

  Any solution which satisfies the governing equation, initial condition and boundary conditions is a solution for that problem.

  Algebraic equation → number

  Differential equation → function

- Uniqueness :

  First, we assume two solutions, and then prove that the two are the same.

- Physically realizable :

  Physical constraint, casaul effect.

  Response occurs after the excitation.

  Examples:

  Case 1: No real solution

  $$\dot{y}^2 + 3 = 0$$

  Case 2: One trivial solution $y(t) = 0$.

  $$\dot{y}^2 + y^2 = 0$$

  Case 3: One solution

  $$\dot{y} + 3y = 0, y(0) = 1$$

  Check the solution ?

- Solution group, undetermined coefficient

- Complete solution=complementary solution(homo. + parti.)

- Singular solution ($y = cx + 2c^2$) with $y(1) = -1/8$

  $$y = xy' + 2(y')^2$$

  $$y(x) = -x^2/8$$

  Nonunique solution

線性微分方程式-解的存在、唯一與合理性？

海大河工系 陳正宗

Definition: normal ODE

A linear differential equation is normal on an interval I if and only if its coefficient functions and its nonhomogeneous term, if it has one, are continuous and the value of leading coefficient is never zero on I.

Existence theorem:

If x_0 is contained in an interval over which the ODE is normal, and if $k_0, k_1, \cdots, k_{n-1}$ are arbitrary real numbers, then there exists exactly one solution $y(x)$ on I such that $y(x_0) = k_0, y'(x_0) = k_1, \cdots, y^{(n-1)}(x_0) = k_{n-1}$.

Solution families:

$y_1(x)$ satisfy n^{th} homogeneous ODE
$y_2(x)$ satisfy n^{th} homogeneous ODE
$\cdots$ satisfy n^{th} homogeneous ODE
$y_n(x)$ satisfy n^{th} homogeneous ODE
$y_p(x)$ satisfy n^{th} nonhomogeneous ODE

Then, the general solution can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

$y_1, y_2 \cdots y_n$ are called complementary solutions.

y_p is called a particular solution.

$y(x)$ is called a general solution.

$c_1, c_2 \cdots c_n$ are determined by the n initial conditions.

Separation of space and time

x:space, t:time, $X_i(x)$:mode, $T_i(t)$: generalized coordinate

$$f(x,t) = X(x)T(t)$$

$$f(x,t) = \sum_{i=0}^{N} X_i(x)T_i(t)$$

$$f(x,t) = \sum_{i=0}^{\infty} X_i(x)T_i(t)$$

Example:

$$f(x,t) = xt$$

$$f(x,t) = sin(x-t) = sin(x)cos(t) - cos(x)sin(t)$$

$$f(x,t) = \frac{1}{\sqrt{1+t^2-2tx}} = \sum_{i=0}^{\infty} t^i P_i(x)$$

$$f(x,t) = \frac{1}{\sqrt{t-x}} =?$$

Example:

$$\frac{dy(x)}{dx} = f(x,y) = X(x)Y(y)$$

$$\int \frac{dy}{Y(y)} = \int X(x)dx + c$$

case 1:

$$\frac{dx}{dt} = t\sqrt{1-x^2}$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int tdt + c \quad \rightarrow \quad x(t) = sin(\frac{t^2}{2}+c)$$

case 2: Escape velocity

$$\ddot{r}(t) = -g\frac{R^2}{r^2}$$

$$\dot{v}(t) = \frac{dv(t)}{dt} = -g\frac{R^2}{r^2}$$

$$\dot{v}(t) = \frac{dv(t)}{dr}\frac{dr}{dt} = -g\frac{R^2}{r^2}$$

$$\dot{v}(t) = \frac{dv(t)}{dr}v = -g\frac{R^2}{r^2}$$

$$v^2 = \frac{2gR^2}{r} + C$$

At time $t = 0$,

$$t = 0, r(0) = R, v(0) = v_0 \quad \rightarrow \quad C = -2gR + v_0^2$$

Escape velocity, $r = \infty, v = 0$ to determine the v_0

$$v_0 = \sqrt{2gR}$$

Clairaut's 微分方程式 奇異解的解法

海大河海系 陳正宗

Clairaut's 微分方程式:

$$y = xy' - \frac{1}{4}y'^2$$

General solution:

$$y = xc - \frac{1}{4}c^2, for\ \ any\ \ c$$

Singular solution $y = y(x)$ by parameter representation

$$x = x(c), y = y(c)$$

Two conditions must be satisfied if $(x(c), y(c))$ is intersection point with the same tangent line

$$y(c) = x(c)\,c - \frac{1}{4}c^2 \tag{1}$$

$$\frac{dy}{dx}\,|_{(x(c),y(c))} = c$$

By considering

$$\frac{dy}{dx}\,|_{(x(c),y(c))} = \frac{dy(c)/dc}{dx(c)/dc}\,|_{(x(c),y(c))} = c \rightarrow y'(c) = cx'(c)$$

Eq.(1) can be differentiated with respect to c, we have

$$y'(c) = x'(c)\,c + x(c) - \frac{1}{2}c$$

Therefore, we have

$$x(c) = \frac{1}{2}c, y(c) = \frac{1}{4}c^2$$

The singular solution is $y = x^2$.

direct solution for the ODE:

Setting $y' = p$, we have

$$y = xp - \frac{1}{4}p^2$$

$$\frac{dy}{dx} = p + xp' - \frac{1}{4}2p'$$

$$p'(x - \frac{1}{2}p) = 0$$

$$p' = 0 \ \ \rightarrow \ \ p(x) = c \ \ \rightarrow y(x) = cx + k_1$$

$$p = 2x \ \ \rightarrow y(x) = x^2 + k_2$$

where k_1 and k_2 can be determined by substituting into Clairaut's equation.

Exercise: $y = xp - e^p$ where $p = y'$. Solve the general solution and singular solution.

Clairant's equation: $y(x) = xy'(x) - \frac{1}{4}\left(y'(x)\right)^2$

我們已知 $y'(x) = c$，i.e. $y(x) = cx + \frac{1}{4}c^2$。**假設此解存在一條包絡線，則直線必和包絡線相切於一點，** $\left(x(c)\ ,\ y(c)\right)$ **（如圖 1），代入原等式我們可得知：**

$$y(c) = c\,x(c) + \frac{1}{4}c^2 \qquad \text{（第 1 式）}$$

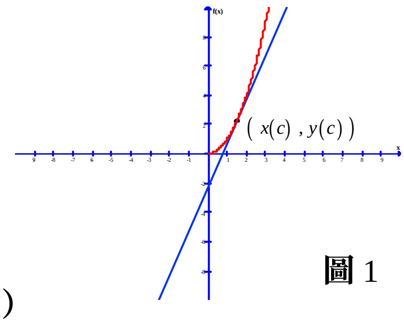


圖 1

將其對 c **微分，我們可知：**

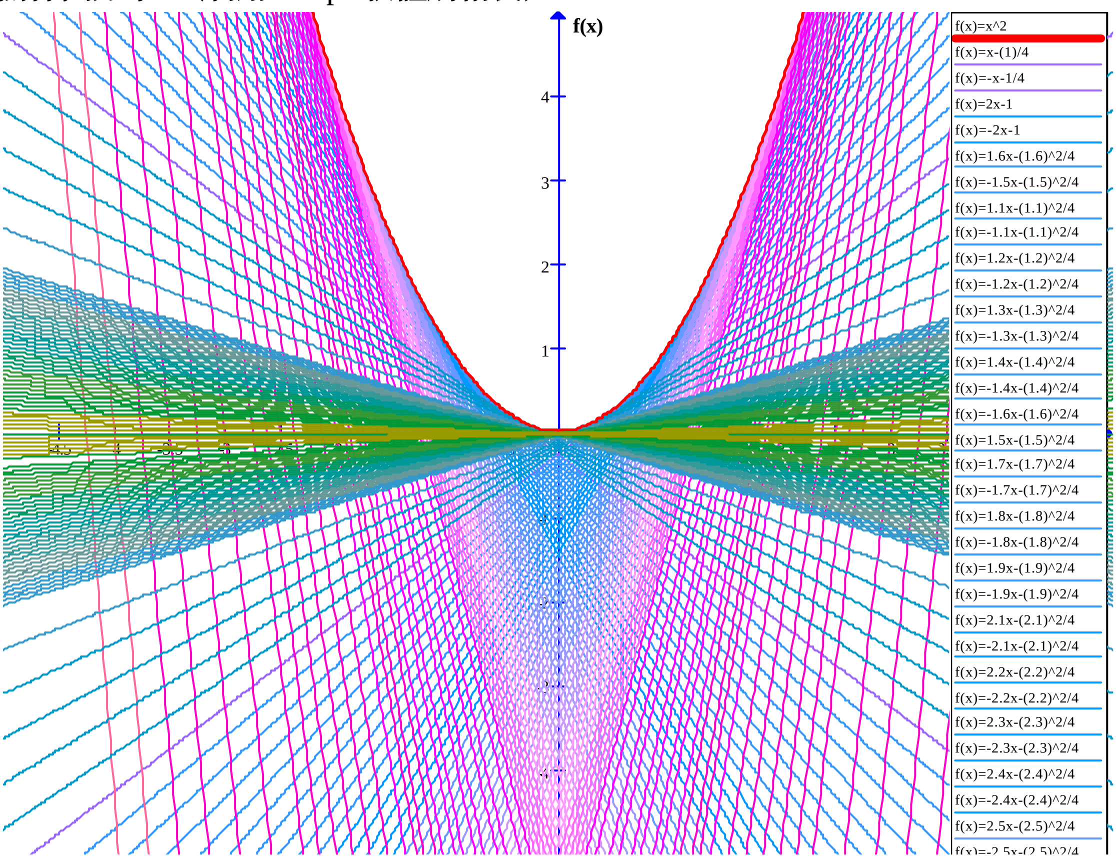
$$y'(c) = c\,x'(c) + x(c) - \frac{1}{2}c \qquad \text{（第 2 式）}$$

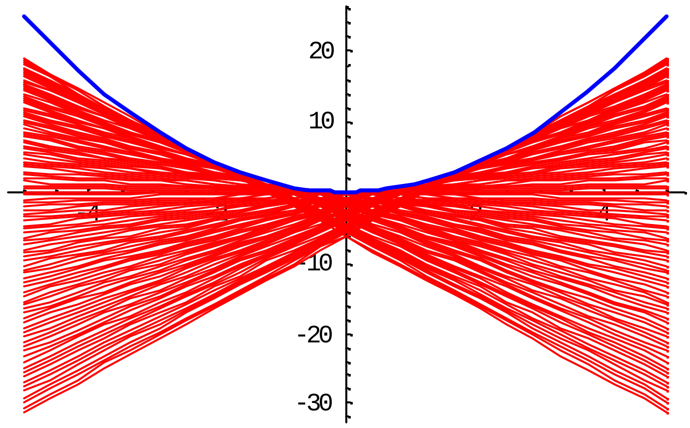
又該點斜率為 $c = \frac{dy}{dx} = \frac{\frac{dy(c)}{dc}}{\frac{dx(c)}{dc}} = \frac{y'(c)}{x'(c)}$，i.e. $y'(c) = c\,x'(c)$ **代入第 2 式，可知**

$x(c) = \frac{1}{2}c$，**代入 1 式，求得** $y(c) = \frac{1}{4}c^2 = \left(\frac{1}{2}c\right)^2 = x^2$，**所以我們可知包絡線為：**

$$y(x) = x^2$$

解的關係圖如下：（利用 Graph 軟體所繪製）





20
10
-10
-20
-30

海大河海系 陳正宗

Bernoulli equation : nonlinear form

$$\dot{y}(x) + a(x)y(x) = f(x)y^n(x)$$

Change of variables: nonlinear transformation

$$z(x) = y^{1-n}(x)$$

Linear ODE : standard form

$$\dot{z}(x) + (1-n)a(x)z(x) = (1-n)f(x)$$

Special case : nonlinear form

$$\dot{y}(x) + a(x)y(x) = f(x)\, y\, ln(y)$$

Change of variables: nonlinear transformation

$$z(x) = ln(y)$$

Linear ODE : standard form

$$\dot{z}(x) + a(x) = f(x)z$$

Ricatti equation : nonlinear form for control theory

$$\dot{y}(x) = P(x)y^2 + Q(x)y + R(x)$$

$$R(x) = 0 \ \rightarrow Bernoulli\ equation$$

$$R(x) \neq 0 \ \rightarrow how\ to\ solve$$

Clairaut equation :

$$y = x\dot{y}(x) + f(\dot{y})$$

Examples :

$$3x\dot{y}(x) + y(x) = x^2y^4(x)$$

$$\dot{y}(x) + y(x) = xy^3(x), y(0) = 1$$

$$\dot{y}(x) + y(x)/x = y^2(x)ln(x), y(1) = 1$$

Determine $a(x), f(x)$ and n.

Determine the solution.

Envelope by a family of geometry unit

海大河工系 陳正宗

Clairaut's 微分方程式:

$$y = xy' - \frac{1}{4}y'^2$$

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$$\frac{dy}{dx}\mid_{(x(c),y(c))} = c$$

By considering

$$\frac{dy}{dx}\mid_{(x(c),y(c))} = \frac{dy(c)/dc}{dx(c)/dc}\mid_{(x(c),y(c))} = c \rightarrow y'(c) = cx'(c)$$

Eq.(1) can be differentiated with respect to c, we have

$$y'(c) = x'(c)\, c + x(c) - \frac{1}{2}c$$

Therefore, we have

$$x(c) = \frac{1}{2}c, y(c) = \frac{1}{4}c^2$$

The singular solution is $y = x^2$.

Mohr-Columb criterion

$$(x-a)^2 + y^2 = r^2(a)$$

$$2(x-a) + 2y = 2r(a)$$

The envelope is a straight line.

Monge cone

$$z - z_0 = p(s)(x - x_0) + q(s)(y - y_0)$$

$$0 = p'(s)(x - x_0) + q'(s)(y - y_0)$$

We have the envelope of Monge cone.

海大河工系 陳正宗

原微分方程式$M(x,y)\,dx+N(x,y)\,dy=0$若非正合時, 仍有可能找到一 個函數$\mu(x,y)$, 使得$\mu M\,dx+\mu N\,dy=0$爲正合型, 即

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\quad \text{但是}\quad \frac{\partial[\mu M]}{\partial y}=\frac{\partial[\mu N]}{\partial x}$$

則稱$\mu(x,y)$爲此微分方程式之積分因子 (Integrating factor).

例如$2y\,dx+x\,dy=0$不是正合方程式, 但乘上x後, 則得到一正合方程式

$$2xy\,dx+x^2\,dy=d(x^2y)=0$$

亦即x是方程式$2y\,dx+x\,dy=0$的一個積分因子。

根據積分因子的定義,$\mu(x,y)$應滿足

$$\frac{\partial[\mu M]}{\partial y}=\frac{\partial[\mu N]}{\partial x}\ \Rightarrow\ \mu\frac{\partial M}{\partial y}+M\frac{\partial \mu}{\partial y}=\mu\frac{\partial N}{\partial x}+N\frac{\partial \mu}{\partial x}$$

$$\text{亦即}\ N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

此爲一階偏微分方程式, 不易求解, 故無實用性可言。只有在下表所列幾種較單純 的情況 —— 即$\mu(x,y)$只爲x, y, $x+y$或xy 的函數, 積分因子才有實用性。

判　別　式	積分因子μ
$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$	$e^{\int f(x)dx}$
$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=f(y)$	$e^{\int f(y)dy}$
$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N-M}=f(x+y)$	$e^{\int f(x+y)\,d(x+y)}$
$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{Ny-Mx}=f(xy)$	$e^{\int f(xy)\,d(xy)}$

積分因子: $dy/dx=a(x)y$ (case 1)
積分因子: x^my^n (Kaplan)

Second order nonlinear ODE:

$$y'' = c\sqrt{1+y'^2}$$

Boundary conditions:

$$y(0) = 0, y'(0) = 0$$

Reducible from second order to first order

$$v = y'$$

$$v\frac{dv}{dy} = c\sqrt{1+v^2}$$

$$\sqrt{v^2+1} = cy+1$$

Solve first order ODE again:

$$y'^2 = (cy+1)^2 - 1$$

Setting new variable u

$$u = cy+1$$

First order ODE change to

$$\frac{du}{\sqrt{u^2-1}} = c\ dx$$

$$cosh^{-1}(u) = cx+k$$

$$cosh(cx+k) = u = cy+1$$

$$k = 0,\ \ since\ \ y(0) = 0$$

Final solution:

$$y(x) = \frac{1}{c}\{cosh(cx) - 1\}$$

海大河工系陳正宗 工數 (一)

存檔:*cable.ctx* 建檔: Nov./1/'98

1.一階線性常微分方程式:$y'+p(x)y=q(x)$之解 $y(x)=y_h+y_p=e^{-\int p(x)dx}\left[c+\int e^{\int p(x)dx}\cdot q(x)\,dx\right]$
2.Bernoulli's方程式:$y'+p(x)y=q(x)y^n \quad (n\neq 0,1)$
3.Riccati's方程式:$y'+p(x)y+q(x)y^2=r(x)$
4.Clairaut's方程式:$y=xy'+q(y')$

1. 一階常微分方程式之通式:$M(x,y)\,dx+N(x,y)\,dy=0$ 或 $y'=f(x,y)$
   解題流程:
   (a) 判斷是否可分離變數? 是否爲齊次方程式?
   (b) 判斷是否爲正合方程式? 嘗試找其積分因子?
   (c) 判斷是否可合併成全微分式?
   (d) 判斷是否爲一階線性、非線性或高階常微分方程式?

2. 一階線性常微分方程式:$y'+p(x)y=q(x)$
   公式:$y(x)=y_h+y_p=e^{-\int p(x)dx}\left[c+\int e^{\int p(x)dx}\cdot q(x)\,dx\right]$

3. 一階非線性常微分方程式
   (a) Bernoulli's 方程式:$y'+p(x)y=q(x)y^n \quad (n\neq 0,1) \;\Rightarrow\;$ 令 $v(x)=y^{1-n}$
   可化爲一階線性常微分方程式
   (b) Riccati's 方程式:$y'+p(x)y+q(x)y^2=r(x) \;\Rightarrow\;$ 令 $y(x)=y_1(x)+v(x)$
   可化爲 Bernoulli's 方程式 (先設法找到一解$y_1(x)$)
   (c) Clairaut's 方程式:$y=xy'+q(y') \;\Rightarrow\;$ 令 $v=y'$
   可得一奇異解及一通解

4. 一階高次及可降階之二階常微分方程式 $\Longrightarrow$ 採用適當的因變數變換

5. 其它相關重點:
   (a) 通解與奇解之關係
   (b) 解之存在與唯一性
   (c) 正交軌跡
   (d) 近似解 —— Picard's 疊代法、微擾法

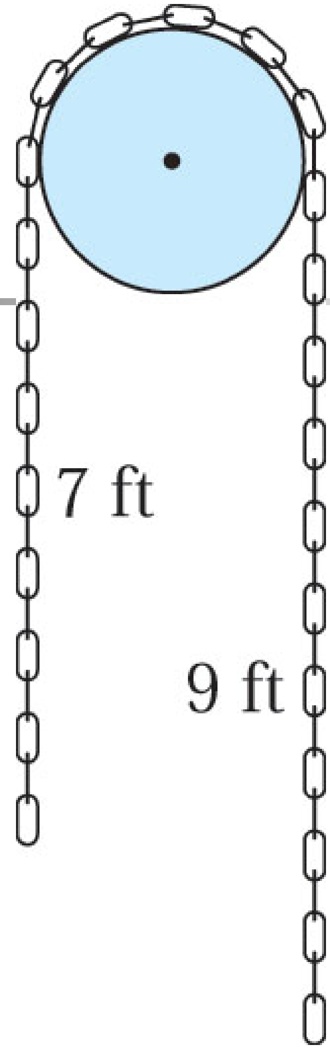


Figure 1.16 ***Chain on a pulley (chain.ppt)***

$$\int \frac{1}{\sqrt{p^2-1}}\,dp = ?$$

※ $\int \frac{1}{\sqrt{p^2-1}}\,dp = \cosh^{-1}(x)$

$$p = \cosh(x)$$

$$\frac{d\cosh(x)}{dx} = \sinh(x)$$

$$\sqrt{p^2-1} = \sinh(x)$$

$$\int \frac{1}{\sqrt{p^2-1}}\,dp = \int \frac{1}{\sinh(x)}\sinh(x)dx = x = \cosh^{-1}(p)$$

※ $\cosh^{-1}(x) = q = \ln(x \pm \sqrt{x^2-1})$

$$\cosh(q) = x$$

$$e^{q} + e^{-q} = 2x$$

$$e^{2q} - 2xe^{q} + 1 = 0$$

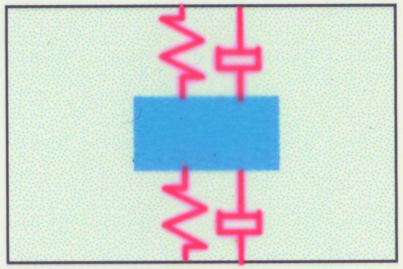
$$e^{q} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^{q} = x \pm \sqrt{x^2-1}$$

$$q = \ln(x \pm \sqrt{x^2-1})$$

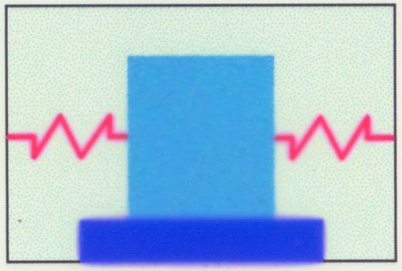
第二章 二階常微分方程式

Casuality in Modelling

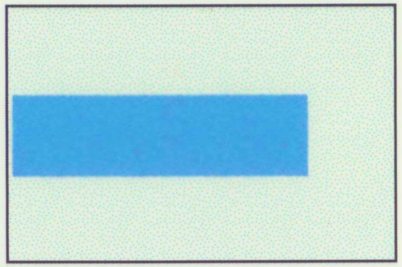


Viscous Damping

Casuality O.K.

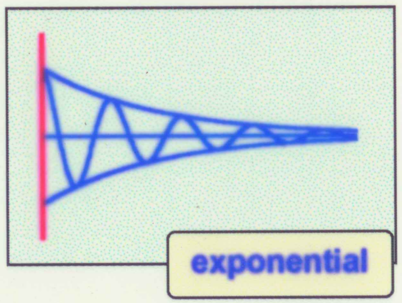


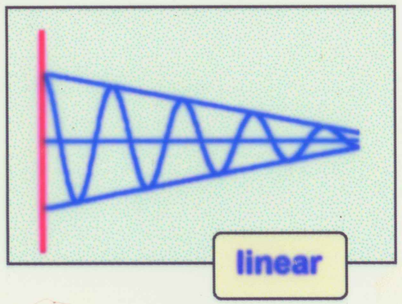
Coulomb Damping

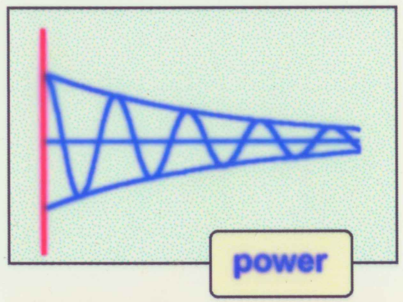


Hysteretic Damping

Casuality ?







S. S. Rao (1990) ?

J. Argyris (1991) ?

$$m\ddot{u} + k(1+i\eta)u = P(t) \longrightarrow m\ddot{u} + h\frac{|u|}{|\dot{u}|}\dot{u} + ku = P(t)$$

二階與高階微分方程式

海大河海系 陳正宗

First order ODE

$$y' + a(x)y = f(x), y(x_0) = y_0$$

Second order ODE

$$y'' + a(x)y' + b(x)y = f(x), y(x_0) = y_0, y'(x_0) = y_1$$

nth order ODE

$$y^n(x) + a_{n-1}(x)y^{n-1}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

initial conditions:

$$y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{n-1}(x_0) = y_{n-1}$$

Homogeneous if $f(x) = 0$, otherwise nonhomogeneous.

Linearity if y_1 and y_2 both satisfy the homogeneous ODE, then $y_1 + y_2$ satisfies the homogeneous ODE.

Existence and uniqueness theorem: if $a_0(x), \cdots a_{n-1}(x), f(x)$ are all continuous on the interval (x_1, x_2).

No. of independent solutions

$$y'' - y = 0$$

Sol.: $y(x) = e^x, e^{-x}, cosh(x), sinh(x)$ all satisfy the ODE

only two conditions to determine the coefficients.

what is wrong ?

Linear independence and dependence

$c_1y_1(x) + c_2y_2(x) = 0 \rightarrow$ only choice of $c_1 = c_2 = 0$.

$c_1y_1(x) + c_2y_2(x) = 0 \rightarrow c_1 \neq 0$ or $c_2 \neq 0$.

Vector space : $(0, 1)$ and $(1, 0)$

Function space : e^x, e^{-x}, $cosh(x)$ and $sinh(x)$.

Complementary solution

$$y''''(x) - y(x) = 0$$

Sol. : $y_h(x) = e^x, e^{-x}, cos(x), sin(x), cosh(x), sinh(x)$

Only four functions are independent to match four conditions.

Particular solution

$$y''''(x) - y(x) = 1$$

Sol. : $y_p(x) = -1$

Any particular solution plus a complementary solution is another particular solution.

Example:

$$u''(x) = sin(x)$$

$$u(x) = a + bx - sin(x)$$

where a and b are determined by boundary conditions or initial conditions.

Extension to integral equation:

$$u(x) = \{u(s)\frac{dU(s,x)}{ds} - u'(s)U(s,x)\} \mid_0^\pi + \int_0^\pi U(s,x)sin(s)ds, 0 < x < \pi$$

where

$$U(s,x) = \frac{1}{2} \mid x - s \mid$$

Complementary sol.:

$$u(x) = \{u(s)\frac{dU(s,x)}{ds} - u'(s)U(s,x)\} \mid_0^\pi$$

Particular sol.:

$$u(x) = \int_0^\pi U(s,x)sin(s)ds = -sin(x) + \pi, 0 < x < \pi$$

初始値問題與邊界値問題

海大河海系 陳正宗

Initial value problem ; ODE with initial conditions at initial of time $t = 0$

$$\frac{dy}{dx} - 2y = 0, y(0) = 2$$

$$\ddot{x} + 5\dot{x} + x(t) = 0, x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

General form: Governing equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y'(x) + a_n(x)y = f(x)$$

Initial conditions

$$y(x_0) = k_0, y'(x_0) = k_1, \cdots, y^{(n-1)}(x_0) = k_{n-1}$$

Boundary value value problem : ODE with boundary conditions at boundary of space $x = 0, l$

string sujected to loading

$$\frac{d^2y}{dx^2} = 1, y(0) = 0, y(1) = 0$$

beam sujected to loading

$$\frac{d^4y}{dx^4} = 1, y(0) = 0, y"(0) = 0, y(1) = 0, y"(1) = 0$$

Sturm-Liouville problem : ODE with boundary conditions at boundary of space $x = a, b$

Governing equation

$$a_0(x)y"(x) + a_1(x)y'(x) + a_2(x)y(x) = f(x)$$

Boundary conditions

$$\alpha_1 y(a) - \alpha_2 y(a) = 0$$

$$\beta_1 y(b) - \beta_2 y(b) = 0$$

where $\alpha_1^2 + \alpha_2^2 \neq 0, \beta_1^2 + \beta_2^2 \neq 0$.

No. of conditions = No. of undeterined coefficient of complementary parts

線性獨立、線性相關與 Wronskian

海大河海系 陳正宗

1. Vector space $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4} \cdots$
2. function space $f_1, f_2, f_3, \cdots$
3. Linear independence for vector space

   $$\mathbf{v_1} = (1, 0), \mathbf{v_2} = (0, 1)$$

4. Linear dependence for vector space

   $$\mathbf{v_1} = (1, 1), \mathbf{v_2} = (2, 2)$$

5. Linear independence for function space

   $$f_1(x) = e^x, f_2(x) = e^{-x}$$

6. Linear dependence for function space

   $$f_1(x) = e^x, f_2(x) = e^{-x}, f_3(x) = cosh(x)$$

7. Linear independence for function space and vector space

   $$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \ \ implies \ \ c_1 = c_2 = c_3 = \cdots = c_n = 0$$

   $$c_1 \mathbf{v}_1(x) + c_2 \mathbf{v}_2(x) + \cdots + c_n \mathbf{v}_n = 0 \ \ implies \ \ c_1 = c_2 = c_3 = \cdots = c_n = 0$$

8. Linear dependence for function space and vector space:

   Functions $f_1, f_2, \cdots f_n$ are linearly dependent on an interval I if and only if there exists $c_1, c_2, \cdots, c_n$ at least one of which is not zero, such that every x in I,

   $$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) + \cdots + c_n f_n(x) = 0$$

9. inner product of vectors, determinant of vectors and Wronskian of function space

   $$\begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix}, \quad \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

海大河海系 陳正宗

1. Given a second order ODE

$$\ddot{x}(t) + 2\xi\omega\dot{x}(t) + \omega^2 x(t) = 0$$

2. General solution

$$x(t) = e^{st}$$

3. s must satisfy

$$s^2 + 2\xi\omega s + \omega^2 = 0$$

4. Two roots are

$$s_1 = (-\xi + \sqrt{\xi^2 - 1})\omega$$

$$s_2 = (-\xi - \sqrt{\xi^2 - 1})\omega$$

5. If $0 < \xi < 1$, two solutions are

$$x_1(t) = x_{1r}(t) + ix_{1i}(t) = e^{-\xi\omega t}cos(\sqrt{1-\xi^2}t) + ie^{-\xi\omega t}sin(\sqrt{1-\xi^2}t)$$

$$x_2(t) = x_{2r}(t) + ix_{2i}(t) = e^{-\xi\omega t}cos(\sqrt{1-\xi^2}t) - ie^{-\xi\omega t}sin(\sqrt{1-\xi^2}t)$$

6. Substituting the two solutions into the ODE, we have

$$\ddot{x}_{1r}(t) + 2\xi\omega\dot{x}_{1r}(t) + \omega^2 x_{1r}(t) + i\{\ddot{x}_{1i}(t) + 2\xi\omega\dot{x}_{1i}(t) + \omega^2 x_{1i}(t)\} = 0 + 0i$$

$$\ddot{x}_{2r}(t) + 2\xi\omega\dot{x}_{2r}(t) + \omega^2 x_{2r}(t) + i\{\ddot{x}_{2i}(t) + 2\xi\omega\dot{x}_{2i}(t) + \omega^2 x_{2i}(t)\} = 0 + 0i$$

7. Two complementary solutions are

$$x_{1c}(t) = e^{-\xi\omega t}cos(\sqrt{1-\xi^2}t)$$

$$x_{2c}(t) = e^{-\xi\omega t}sin(\sqrt{1-\xi^2}t)$$

M, C, K System

$$m\ddot{x} + c\dot{x} + kx = 0 \Leftrightarrow \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0$$

$$\frac{c}{m} = 2\xi\omega \rightarrow \xi : damping\ ratio\ (\text{阻尼比})$$

$$\frac{k}{m} = \omega^2 \rightarrow \omega : natural\ \ frequency\ (\text{自然頻率})$$

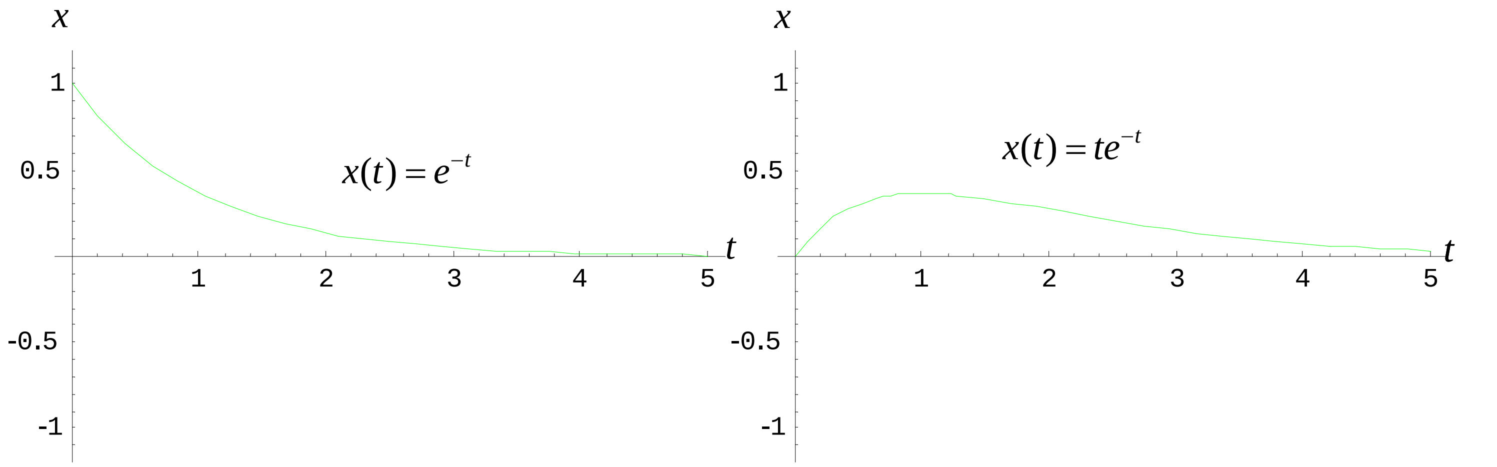
$\xi > 0$ $physically\ realizable\ (diverge)$，$\xi < 0$ $physically\ unrealizable\ (converge)$

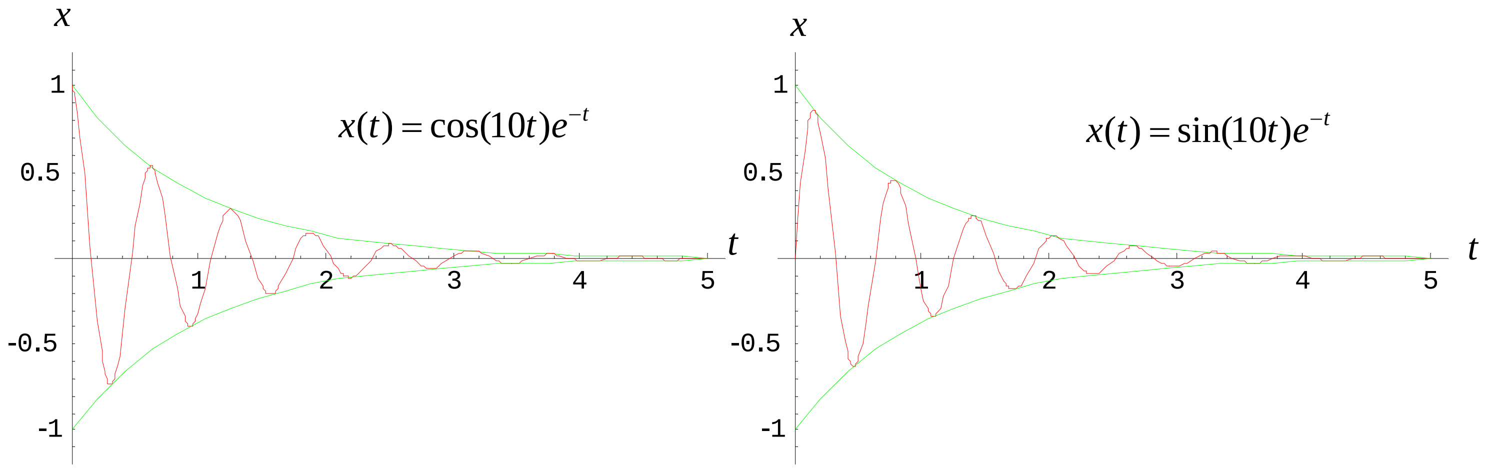
$sub - damping$ $critical\ damping$ $over - damping$

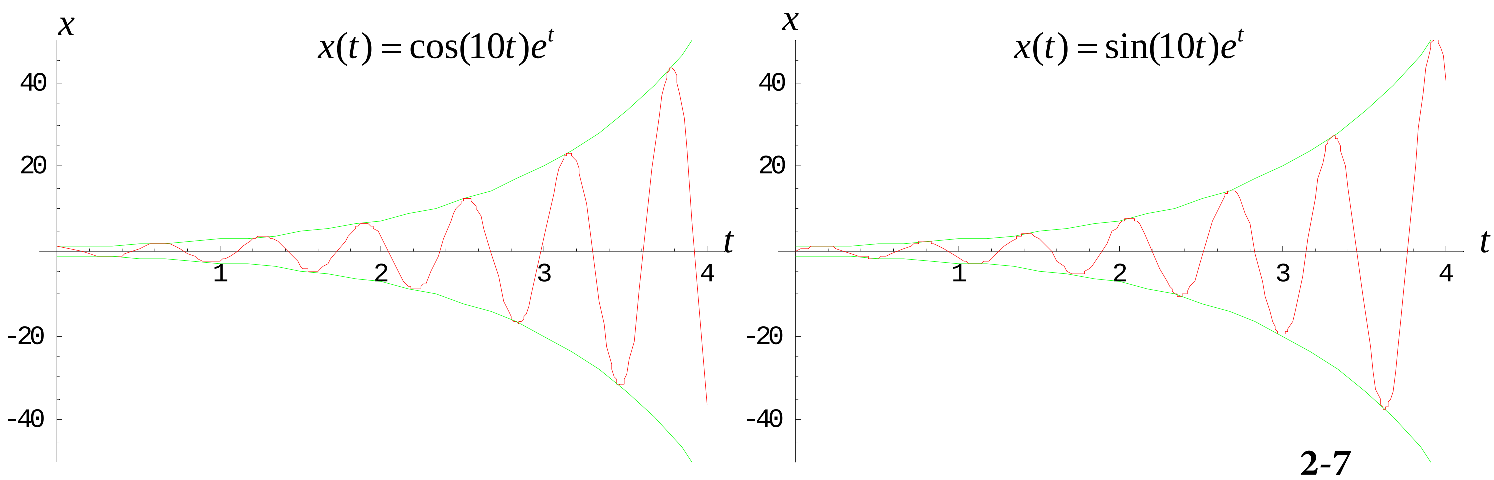
$$0.99...9 < \xi < 1.00...1$$

↓

$Oscillation$ 分水嶺（重根時）







1. Step 1: given one complementary solution, y_1.

2. Step 2: solve another complementary solution $y_2 = y_1 u_1$.

3. Step 3: solve another particular solution $y_p = y_1 v_1 + y_2 v_2$.

4. Example :

$$x^2 y''(x) - 4xy' - 6y = -6 \tag{1}$$

(a). Assume the $y = x^n$ for the complementary solution, determine n. (5%)
(b). If $y_1(x) = \frac{1}{x}$ is one of the complementary solution, please determine the other one $y_2(x)$ by method of variations of parameters, $y_2(x) = y_1(x)u_1(x)$. Please find $u_1(x)$. (5%)
(c). Solve the particular solution by $y_p(x) = y_1(x)v_1(x) + y_2(x)v_2(x)$, where

$$y_1 v_1' + y_2 v_2' = 0$$

$$y_1' v_1' + y_2' v_2' = \frac{-6}{x^2}$$

Please determine v_1, v_2 and y_p. (5%)
(d). By changing variable, $x = e^t$ and $y(x) = y(e^t) = Y(t)$, then determine the ODE for $Y(t)$ and solve $Y(t)$ and $y(x)$. (5%)
(e). By taking the Laplace transform twice with respect to Eq.(3), derive the results. (5%)

海大河工系 陳正宗

Statement of problem

Given two solutions y_1 and y_2 satisfy

$$a_0(x)y''(x)+a_1(x)y'(x)+a_2(x)y(x)=0$$

Find a solution $y_p(x)$ satisfy

$$a_0(x)y''(x)+a_1(x)y'(x)+a_2(x)y(x)=f(x) \tag{1}$$

Review of linear algebra:

Given

$$a_1x+b_1y=c_1$$

$$a_2x+b_2y=c_2$$

The solution of (x,y) is

$$x=\frac{\Delta_1}{\Delta}$$

$$y=\frac{\Delta_2}{\Delta}$$

where

$$\Delta=a_1b_2-a_2b_1$$

$$\Delta_1=c_1b_2-c_2b_1$$

$$\Delta_2=a_1c_2-a_2c_1$$

ODE:

$$a_0(x)y_1''(x)+a_1(x)y_1'(x)+a_2(x)y_1(x) = 0 \tag{2}$$

$$a_0(x)y_2''(x)+a_1(x)y_2'(x)+a_2(x)y_2(x) = 0 \tag{3}$$

Setting

$$y_p=u_1y_1+u_2y_2 \tag{4}$$

$$y_p'=u_1'y_1+u_2'y_2+u_1y_1'+u_2y_2' \tag{5}$$

To solve $y_p(x)$ is changed to solve u_1 and u_2.

Two degrees of freedom, u_1 and u_2, must be determined. By setting the first constraint,

$$u_1' y_1 + u_2' y_2 = 0 \tag{6}$$

Eq.(5) can be reduced to

$$y_p' = u_1 y_1' + u_2 y_2' \tag{7}$$

Differentiating x again, we have

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' \tag{8}$$

Substituting Eq.(8) and (7) into Eq.(1), we have

$$a_0(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'') + a_1(u_1 y_1' + u_2 y_2') + a_2(u_1 y_1 + u_2 y_2) = f(x) \tag{9}$$

Eq.(9) can be reformulated to

$$\begin{aligned} & u_1(a_0(x) y_1''(x) + a_1(x) y_1'(x) + a_2(x) y_1(x)) \\ + \; & u_2(a_0(x) y_2''(x) + a_1(x) y_2'(x) + a_2(x) y_2(x)) \\ + \; & a_0(u_1' y_1' + u_2' y_2') = f(x) \end{aligned} \tag{10}$$

Since y_1 and y_2 are solutions of homogeneous ODE, we have

$$u_1' y_1' + u_2' y_2' = \frac{f(x)}{a_0} \tag{11}$$

Two equations are summarized

$$y_1 u_1' + y_2 u_2' = 0 \tag{12}$$

$$y_1' u_1' + y_2' u_2' = \frac{f(x)}{a_0} \tag{13}$$

Solve u_1' and u_2' first, we have

$$u_1' = \frac{W_1}{W(y_1, y_2)}$$

$$u_2' = \frac{W_2}{W(y_1, y_2)}$$

where $W(y_1, y_2)$ is Wronskian determined by

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$W_1 = -y_2 f(x)/a_0(x)$$

$$W_2 = y_1 f(x)/a_0(x)$$

Since y_1 and y_2 are complementary solutions for ODE, we have

$$y_1''(x) + a(x)y_1'(x) + b(x)y_1(x) = 0 \tag{1}$$
$$y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) = 0 \tag{2}$$

Eq.(1) $\times$ y_2 - Eq.(2) $\times$ y_1, we have

$$\frac{d}{dx}\{y_1y_2' - y_1'y_2\} + a(x)\{y_1y_2' - y_1'y_2\} = 0$$

By setting the Wronskian

$$W(x) = W(y_1, y_2) = y_1y_2' - y_1'y_2$$

$W(x)$ satisfies the following first ODE

$$W'(x) + a(x)W(x) = 0$$

The solution is

$$W(x) = ke^{-\int a(x)dx}$$

Without loss of generality, given two degrees of freedoms, u_1 and u_2, must be determined. By setting the first constraint,

$$u_1'y_1 + u_2'y_2 = c \tag{3}$$

we have

$$u_1'y_1' + u_2'y_2' = f(x) - a(x)\ c \tag{4}$$

where c is arbitrary constant. Two equations are summarized

$$y_1u_1' + y_2u_2' = c \tag{5}$$
$$y_1'u_1' + y_2'u_2' = f(x) - a(x)\ c \tag{6}$$

Solve u_1' and u_2' first, we have

$$u_1' = \frac{-f(x)y_2}{W(y_1, y_2)} + c(\frac{y_2' + ay_2}{W})$$

$$u_2' = \frac{f(x)y_1}{W(y_1, y_2)} - c(\frac{y_1' + ay_1}{W})$$

The two additional terms containg c are present. It is interesting to find that

$$u_1' = c(\frac{y_2' + ay_2}{W})(\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}}) = c(y_2e^{\int a(x)dx})'$$

$$u_2; c(\frac{y_1' + ay_1}{W})(\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}}) = c(y_1e^{\int a(x)dx})'$$

They can be cancelled each other.

使用變數異動法求解特解(通式)

陳正宗 李家瑋 賴芝亭 Nov. 26, 2007

$$y'' + a(x)y' + b(x)y = f(x)$$
$$y_c = c_1 y_1 + c_2 y_2$$

$$\Rightarrow y_1'' + a(x)y_1' + b(x)y_1 = 0 \quad \text{---(1)}$$

$$y_2'' + a(x)y_2' + b(x)y_2 = 0 \quad \text{---(2)}$$

$$(1) * y_2 - (2) * y_1 = 0$$

$$y_1'' y_2 + a(x)y_1' y_2 + b(x)y_1 y_2 - y_2'' y_1 - a(x)y_2' y_1 - b(x)y_1 y_2 = 0$$

$$y_1 y_2'' - y_1'' y_2 + a(x)\left[y_1 y_2' - y_1' y_2\right] = 0$$

$$w'(x) + a(x)w(x) = 0$$

$$\Rightarrow I = e^{\int a(x)dx}$$

$$\therefore w(x) = ke^{-\int a(x)dx} \qquad \Rightarrow w(x)e^{-\int a(x)dx} = k$$

令 $y_p = u_1 y_1 + u_2 y_2$

$$y_p' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$$

$$c(x) = u_1' y_1 + u_2' y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + c(x)$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + c'(x)$$

$$\Rightarrow$$

$$u_1{}' y_1{}' + u_2{}' y_2{}' + u_1 y_1{}'' + u_2 y_2{}'' + c'(x) + a(x)\left[u_1 y_1{}' + u_2 y_2{}' + c(x)\right] + b(x)[u_1 y_1 + u_2 y_2] = f(x)$$

$$u_1\left[y_1{}'' + a(x)y_1{}' + b(x)y_1\right] + u_2\left[y_2{}'' + a(x)y_2{}' + b(x)y_2\right]$$

$$+ u_1{}' y_1{}' + u_2{}' y_2{}' + a(x)c(x) + c'(x) = f(x)$$

$$\therefore u_1{}' y_1{}' + u_2{}' y_2{}' = f(x) - a(x)C(x) - c'(x)$$

$$u_1{}' y_1 + u_2{}' y_2 = c(x) \qquad \text{--(1)}$$

$$u_1{}' y_1{}' + u_2{}' y_2{}' = f(x) - a(x)c(x) - c'(x) \qquad \text{--(2)}$$

$$(1) * y_2{}' - (2) * y_1{}'$$

$$\Rightarrow u_1{}' y_1 y_2{}' - u_1{}' y_1{}' y_2 = c(x)y_2{}' - f(x)y_2 + a(x)c(x)y_2 + c'(x)y_2$$

$$u_1{}' w(x) = -f(x)y_2 + c(x)\left[y_2{}' + a(x)y_2\right] + c'(x)y_2$$

$$u_1{}' = \frac{-f(x)y_2}{w(x)} + c(x)\frac{\left[y_2{}' - a(x)y_2\right]}{w(x)} + \frac{c'(x)y_2}{w(x)}$$

$$(2) * y_1 - (1) * y_1{}'$$

$$\Rightarrow u_2{}' y_1 y_2{}' - u_2{}' y_1{}' y_2 = f(x)y_1 - a(x)C(x)y_1 - c'(x)y_1 - c(x)y_1{}'$$

$$u_2{}' w(x) = f(x)y_1 - c(x)\left[y_1{}' + a(x)y_1\right] - c'(x)y_1$$

$$u_2{}' = \frac{f(x)y_1}{w(x)} - c(x)\frac{\left[y_1{}' + a(x)y_1\right]}{w(x)} - \frac{c'(x)y_1}{w(x)}$$

$$u_1{}' = \frac{-f(x)y_2}{w(x)} + c(x)\frac{\left[y_2{}' - a(x)y_2\right]}{w(x)}\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}} + \frac{c'(x)y_2}{w(x)}\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}}$$

$$u_1{}' = \frac{-f(x)y_2}{w(x)} + \frac{c(x)}{k}\left[y_2 e^{\int a(x)dx}\right]' + \frac{1}{k}c'(x)y_2 e^{\int a(x)dx}$$

$$u_2{}' = \frac{f(x)y_1}{w(x)} - c(x)\frac{\left[y_1{}' - a(x)y_1\right]}{w(x)}\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}} - \frac{c'(x)y_1}{w(x)}\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}}$$

$$u_2{}' = \frac{f(x)y_1}{w(x)} - \frac{c(x)}{k}\left[y_1 e^{\int a(x)dx}\right]' - \frac{1}{k}c'(x)y_1 e^{\int a(x)dx}$$

$\therefore$

$$u_1 = \int \frac{-f(x)y_2}{w(x)}dx + \frac{1}{k}\int c(x)d\left[y_2 e^{\int a(x)dx}\right] + \frac{1}{k}\int\left[c'(x)y_2 e^{\int a(x)dx}\right]dx$$

$$u_1 = \int \frac{-f(x)y_2}{w(x)}dx + \frac{1}{k}c(x)y_2 e^{\int a(x)dx} - \frac{1}{k}\int\left[c'(x)y_2 e^{\int a(x)dx}\right]dx + \frac{1}{k}\int\left[c'(x)y_2 e^{\int a(x)dx}\right]dx$$

$$u_1 = \int \frac{-f(x)y_2}{w(x)}dx + \frac{1}{k}c(x)y_2 e^{\int a(x)dx}$$

$$u_2 = \int \frac{f(x)y_1}{w(x)}dx - \frac{1}{k}\int c(x)d\left[y_1 e^{\int a(x)dx}\right] - \frac{1}{k}\int\left[c'(x)y_1 e^{\int a(x)dx}\right]$$

$$u_2 = \int \frac{f(x)y_1}{w(x)}dx - \frac{1}{k}c(x)y_1 e^{\int a(x)dx} + \frac{1}{k}\int\left[c'(x)y_1 e^{\int a(x)dx}\right]dx - \frac{1}{k}\int\left[c'(x)y_1 e^{\int a(x)dx}\right]dx$$

$$u_2 = \int \frac{f(x)y_1}{w(x)}dx - \frac{1}{k}c(x)y_1 e^{\int a(x)dx}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left[\int \frac{-f(x) y_2}{w(x)} dx + \frac{1}{k} c(x) y_2 e^{\int a(x)dx} \right] y_1 + \left[\int \frac{f(x) y_1}{w(x)} dx - \frac{1}{k} c(x) y_1 e^{\int a(x)dx} \right] y_2$$

$$= y_1 \int \frac{-f(x) y_2}{w(x)} dx + y_2 \int \frac{f(x) y_1}{w(x)} dx$$

∴跟 $c(x)$ 爲多少無關，因此用 0 最方便

Given y_1 is one complementary solution for ODE,

$$y_1''(x) + a(x)y_1'(x) + b(x)y_1(x) = 0 \quad (1)$$

solve another complementary solution y_2. By setting the Wronskian

$$W(x) = W(y_1, y_2) = y_1 y_2' - y_1' y_2,$$

$W(x)$ satisfies the following first ODE

$$W'(x) + a(x)W(x) = 0$$

The solution is

$$W(x) = ke^{-\int a(x)dx}$$

Therefore, we have first order ODE for y_2 as follows:

$$y_2' - \frac{y_1'}{y_1} y_2 = \frac{k}{y_1} e^{-\int a(x)dx} \quad (2)$$

Example:

$$y'' + 3y' = 2y = 0$$

Sol: $y_1 = e^{-x}$, $y_2(x)$ satisfies

$$y_2' - \frac{-e^{-x}}{e^{-x}} y_2 = \frac{k}{e^{-x}} e^{-\int 3dx}$$

$$y_2' + y_2 = ke^{-2x}$$

$$y_2 = ce^{-x} + Ke^{-2x}$$

Exercise:

$$x^2 y''(x) - 4xy' - 6y = -6, \quad (3)$$

(a) if $y_1(x) = \frac{1}{x}$ is one of the complementary solution, solve y_2 using Wronskian.

Wronskian 有何用？

(1)求獨立或相依性

$$\begin{vmatrix} y_1 & y_2 \cdots\cdots\cdots y_n \\ y_1' & y_2' \cdots\cdots\cdots y_n' \\ y_1^{(n-1)} & y_2^{(n-1)} \cdots\cdots y_n^{(n-1)} \end{vmatrix} = \begin{cases} 0 & \text{，相依} \\ nonzero & \text{，獨立} \end{cases}$$

(2)由一補算另一補

$$y'' + p(x)y' + q(x)y = 0$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_1' \end{vmatrix}$$

$$W' + p(x)W = 0 \rightarrow W \rightarrow y_2(given \;\; y_1)$$

(3)由兩補算一特

$$y'' + p(x)y' + q(x)y = f(x)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f \end{cases} \rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$\rightarrow u_1', u_2' \rightarrow u_1, u_2 \rightarrow y_p = y_1 u_1 + y_2 u_2$$

待定係數法-求特解

海大河海系 陳正宗

待定係數法 n階常係數方程式

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

的非齊性解y_p必與$f(x)$有某種關係。所以, 我們可根據$f(x)$的形式及微分關係, 假設一個含有若干未知係數之形式的y_p, 然後代入原式中以決定這些係數, 從而求 出y_p。此法稱爲待定係數法 (method of undetermined coefficients).

以待定係數法求特解時, 常用的y_p假設形式如下表:

$f(x)$	$y_p(x)$
x^n	$k_n x^n + k_{n-1} x^{n-1} + \cdots + k_1 x + k_0$
e^{ax}	ke^{ax}
$\sin ax$	$k_1 \cos ax + k_2 \sin ax$
$\cos ax$	$k_1 \cos ax + k_2 \sin ax$
$x^n e^{ax} \sin ax$	$e^{ax} \{\cos ax\,(k_n x^n + k_{n-1} x^{n-1} + \cdots + k_1 x + k_0) +$ $\sin ax\,(\ell_n x^n + \ell_{n-1} x^{n-1} + \cdots + \ell_1 x + \ell_0)\}$
$x^n e^{ax} \cos ax$	$e^{ax} \{\cos ax\,(k_n x^n + k_{n-1} x^{n-1} + \cdots + k_1 x + k_0) +$ $\sin ax\,(\ell_n x^n + \ell_{n-1} x^{n-1} + \cdots + \ell_1 x + \ell_0)\}$

Examples: (待定係數法) 解$y'' + 9y = \cos 3x + 2xe^x$ (工技化工)

由特徵方程式$\lambda^2 + 9 = 0$之根爲$\lambda = \pm 3i$, 可得齊性解爲

$$y_h(x) = c_1 \cos 3x + c_2 \sin 3x$$

根據待定係數法, 設非齊性解

$y_p(x) = x(a\cos 3x + b\sin 3x) + (kx + \ell)e^x$.

代入原式, 得

$$\begin{aligned} &\{[-6a\sin 3x + 6b\cos 3x - 9ax\sin 3x - 9bx\cos 3x + 2ke^x + (kx+\ell)e^x] \\ &\quad + 9x(a\cos 3x + b\sin 3x) + 9(kx+\ell)e^x\} \\ &= 6b\cos 3x - 6a\sin 3x + 10kxe^x + (2k + 10\ell)e^x \\ &= \cos 3x + 2xe^x \end{aligned}$$

由上式可得$a = 0$, $b = \frac{1}{6}$, $k = \frac{1}{5}$, $\ell = -\frac{1}{25}$, 亦即

$$y_p = \frac{x}{6}\sin 3x + (\frac{x}{5} - \frac{1}{25})$$

於是通解 $y = y_h + y_p = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{6}\sin 3x + (\frac{x}{5} - \frac{1}{25})$

八十二年專門職業及技術人員高等考試試題　　高：11－2　全一頁

類　　科：結構工程技師
科　　目：結構動力分析
考試時間：二小時

一、一懸臂梁如圖1所示，右端有一集中質量m，假設梁之質量不計，（25分）
㈠試求自然週期。
㈡當集中質量m停留在梁上有一段時間後，有一半質量掉落，試求右端之位移狀況（假設阻尼比爲0.01）。

二、一簡支梁如圖2所示，單位長度之質量爲m，上面承受一個等速移動之載重P，假設梁之阻尼比爲零及只考慮第一振態之影響，試推導梁中央位移方程式，並求共振速度。（25分）

三、一結構系統如圖3所示，假設O點之轉角θ爲位移自由度，試建立運動方程式並求其自然週期。（25分）

四、一建築結構如圖4所示，V_1及V_2代表位移自由度，試求
㈠質量矩陣及勁度矩陣。
㈡試以衝量近似法（Approximate－impulse method）求各樓層之最大位移。（25分）

圖1

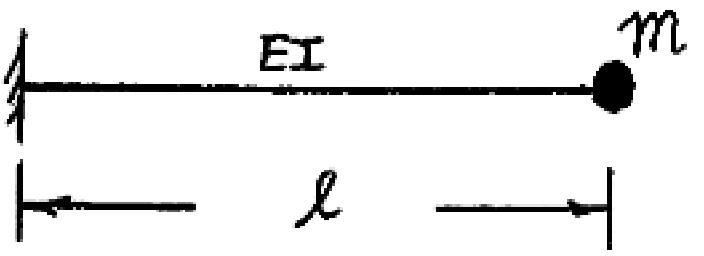


圖2

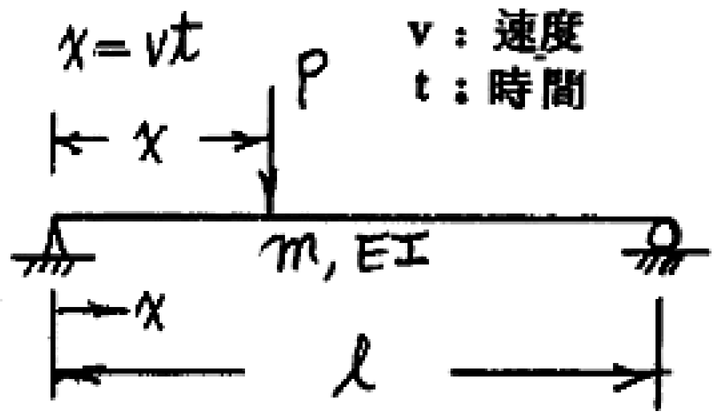


圖3

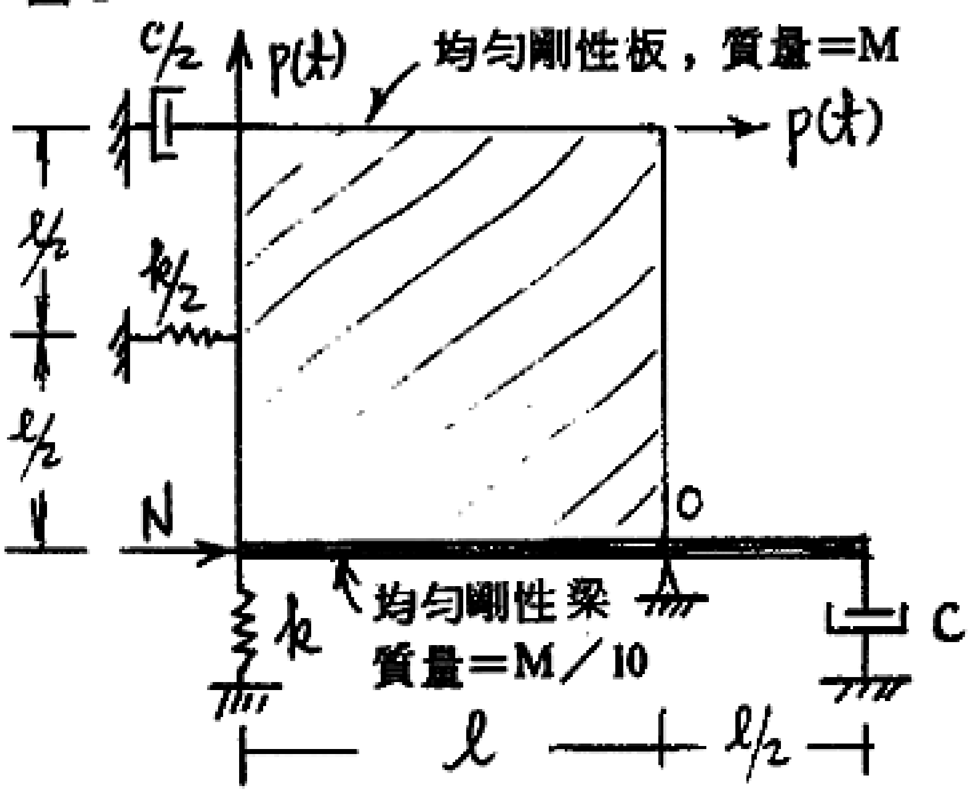
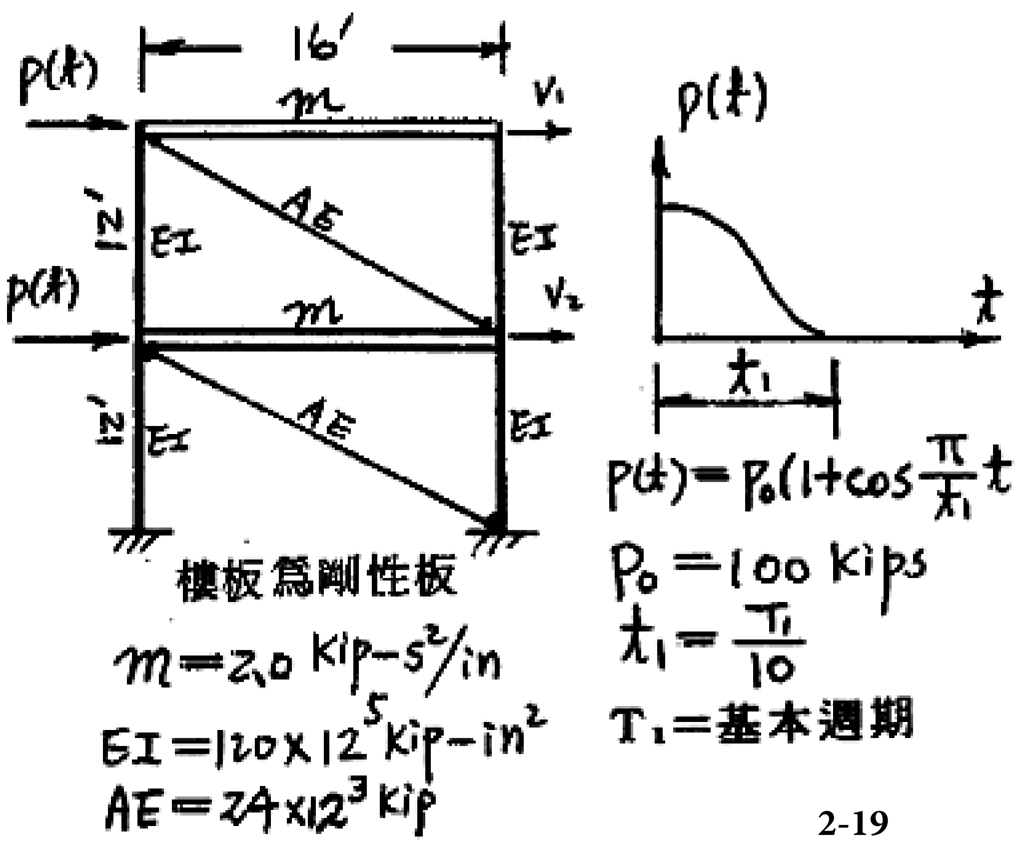


圖4



偏微分方程

第一次群體作業

班級：河工 3B
學號：B94520143
姓名：李家瑋
日期：2008/03/26

群體作業[1]，3/26 交　學號:B94520143　姓名:李家瑋

1. 對於彈簧系統 $y''+cy'+y=0 \quad y(0)=1,\ y(0)=0$

請解出 $c=2.5, c=2, c=1$ 時的 $y(t)$。

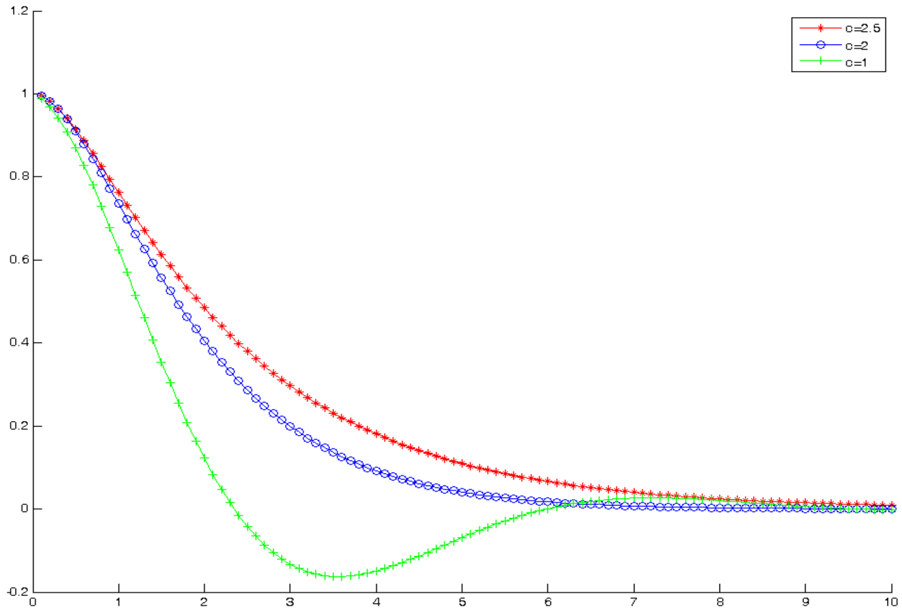
Sol. (1) $c=2.5 \Rightarrow y(t)=\frac{4}{3}e^{-\frac{1}{2}t}-\frac{1}{3}e^{-2t}$

(2) $c=2 \Rightarrow y(t)=e^{-t}+te^{-t}$

(3) $c=1 \Rightarrow y(t)=e^{-\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}t)+\frac{1}{3}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)$

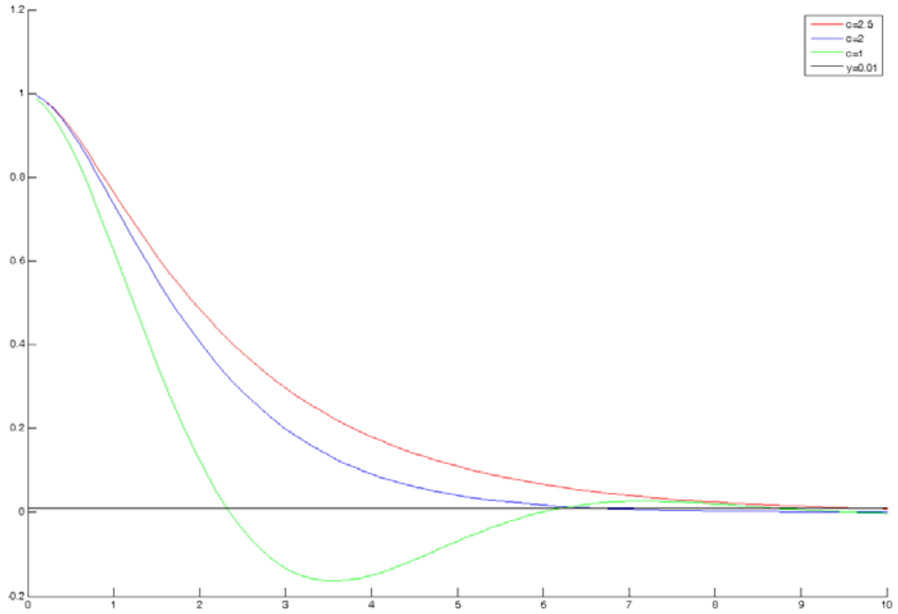
2. 對於 1. 的三種解，請應用程式計算 $y(t)$ 隨著 t 的變化，並將其畫出。

$$\Delta t=0.1,\ N=100$$



遇到複數的問題時，一開始照樣去運算，只是最後畫圖時只取實部的部份，而虛部的部分不要。

3. 由 2. 的解答找出何時振動的振幅小於0.010. 11，即 $|y(t)| < 0.01$。



畫一條 $y = 0.01$ 的直線，看 t 等於何時，其振幅會小於0.01。

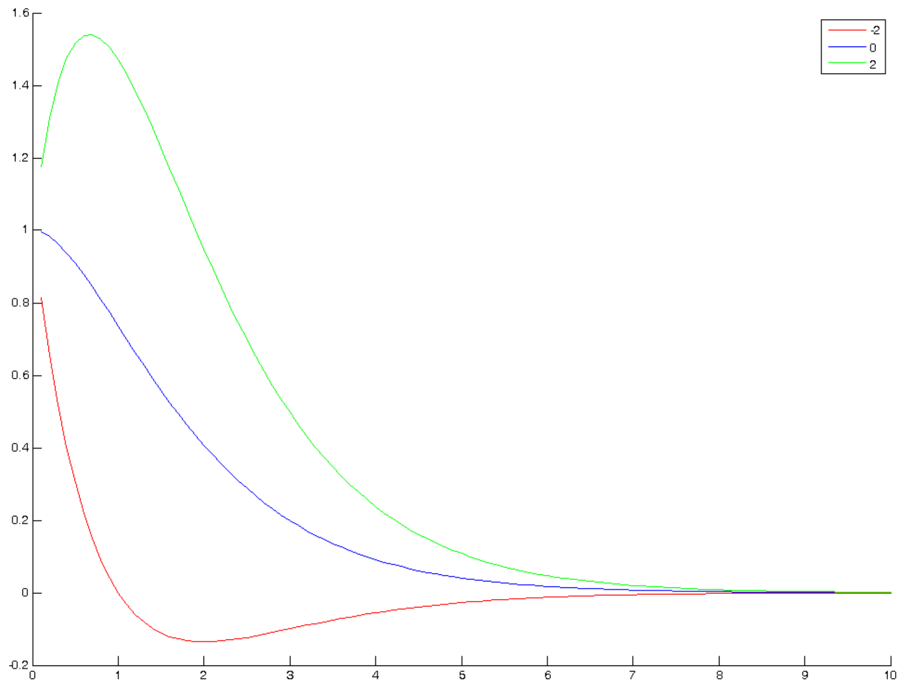
(1)當 $c = 2.5$ 時，約在9.8秒後其振幅開始會小於0.01。

(2)當 $c = 2$ 時，約在6.7秒後其振幅開始會小於0.01。

(3)當 $c = 2.5$ 時，約在2.5~6.2秒後其振幅小於0.01，但6.2秒過後振幅會大於0.01，直到8.7秒後其振幅又小於0.01。

群體作業[1]，3/26 交　　學號:B94520143　　姓名:李家瑋

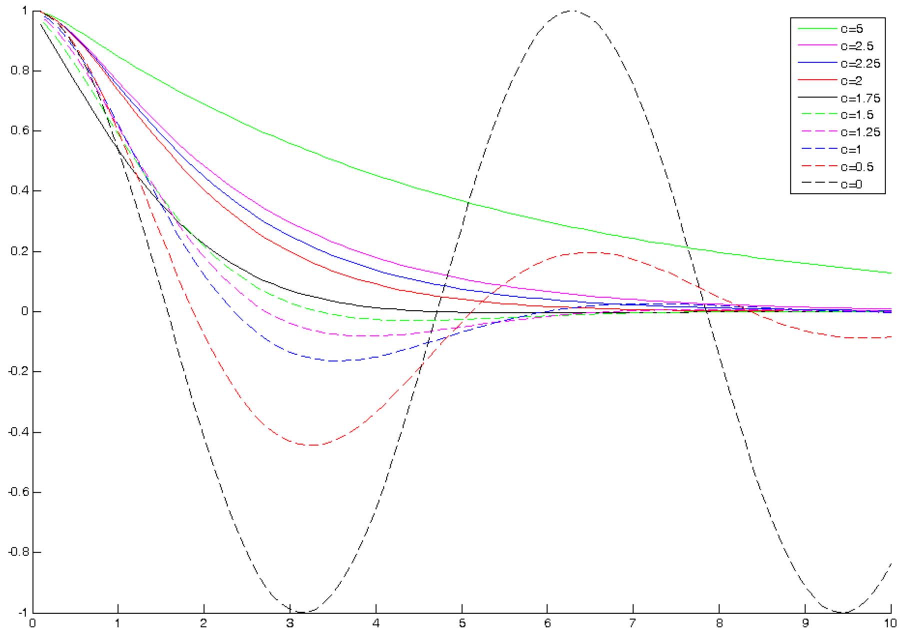
4. 對於 $c=2$，請計算 $y'(0)=2$ 與 $y'(0)=-2$ 的解答，並將其與 1. 中 $y'(0)=0$ 的解一起畫出來比較[比較 $y(t)$ 隨著 t 的變化]。



由圖可以發現，因為初速度(初始條件)給的不同，造成了一開始振動的振幅不一樣，但最後還是會隨時間增加而趨近於0。

5. 回到 1. 的解，可以發現當 c 不同時，彈簧振動的模式也不相同，請應用或修改您針對 1. 與 2. 所撰寫的程式，計算以下所有 c 值的 $y(t)$ 值，並將其畫在一張圖上，觀察探討振動模式的改變，寫下貴組的討論。

$c = \{5.0, 2.5, 2.25, 2.0, 1.75, 1.50, 1.25, 1.0, 0.5, 0.0\}$



由圖可以發現，當 c 值愈來愈小時，其振動的的振幅隨 c 值變小而變大，但最後還是會隨時間增加而趨近於0，除了 $c=0$ 時為簡諧運動，振幅成週期性的變化。

程式碼

```
clear
t=sym('t')
delta=0.1;
c=1;
s=1;
v=0;
if c^2-4>0
    L1=(-c+(c^2-4)^0.5)/2;
    L2=(-c-(c^2-4)^0.5)/2;
    c1=det([s,1;v,L2])/det([1,1;L1,L2]);
    c2=det([1,s;L1,v])/det([1,1;L1,L2]);
    Y=c1*exp(L1*t)+c2*exp(L2*t)
    for N=1:100
    X(N)=N*delta;
    Y1(N)=c1*exp(L1*N*delta)+c2*exp(L2*N*delta);
    end
elseif c^2-4==0
    L1=-c/2;
    c1=det([s,0;v,1])/det([1,0;L1,1]);
    c2=det([1,s;L1,v])/det([1,0;L1,1]);
    Y=c1*exp(L1*t)+c2*t*exp(L1*t)
    for N=1:100
    X(N)=N*delta;
    Y1(N)=c1*exp(L1*N*delta)+c2*N*delta*exp(L1*N*delta);
    end
elseif c^2-4<0
    L1=(-c+i*(4-c^2)^0.5)/2;
    L2=(-c-i*(4-c^2)^0.5)/2;
    c1=det([s,0;v,1])/det([1,0;-c/2,1]);
    c2=det([1,s;-c/2,v])/det([1,0;-c/2,1]);
Y=c1*exp(-c/2*t)*cos(((4-c^2)^0.5)*t/2)+c2*exp(-c/2*t)*sin(((4-c^2)^0.
5*t)/2)
    for N=1:100
    X(N)=N*delta;
Y1(N)=c1*exp(-c/2*N*delta)*cos(((4-c^2)^0.5)*N*delta/2)+c2*exp(-c/2*N*
delta)*sin(((4-c^2)^0.5*N*delta)/2);
    end
end
plot(X,Y1)
```

偏微分方程式

群體作業[1]

本次作業相關內容請參考課本 2.5 節

1. 對於彈簧系統

$$y'' + cy' + y = 0 \qquad y(0) = 1, \quad y'(0) = 0$$

   請解出 c=2.5、c=2 以及 c=1 時的 y(t)；

2. 對於 1.的三種解，請應用程式計算 y(t)隨著 t 的變化，並將其畫出；

   提示：

   A. 設定一Δt，計算 y(Δt)、y(2Δt)、y(3Δt) …y(NΔt) 、y(Δt)，並將其畫出

   B. 請研究在程式化時，如何處理複數的問題

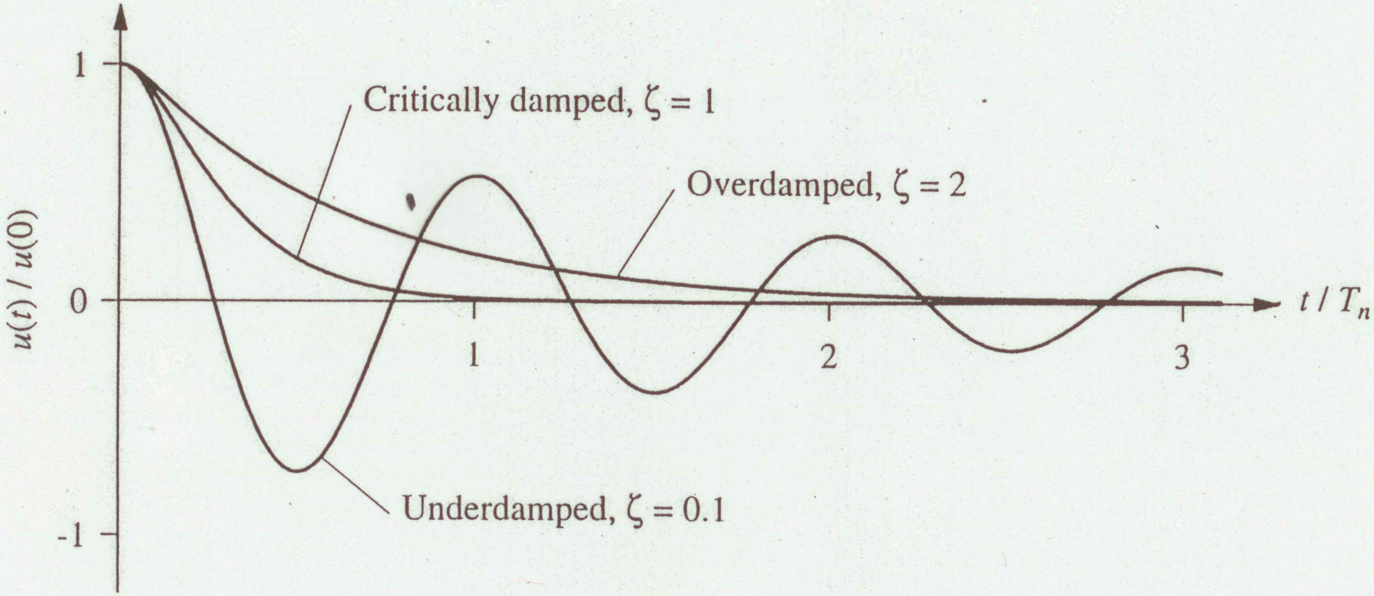
3. 由 2.的解答找出何時振動的振幅小於 0.01，即|y(t)|<0.01；

4. 對於 c=2，請計算 $y'(0) = 2$ 與 $y'(0) = -2$ 的解答，並將其與 1.中 $y'(0) = 0$ 的解一起畫出來比較[比較 y(t)隨著 t 的變化]；

5. 回到 1.的解，可以發現當 c 不同時，彈簧振動的模式也不相同，請應用或修改您針對 1.與 2.所撰寫的程式，計算以下所有 c 值的 y(t)值，並將其畫在一張圖上，觀察探討振動模式的改變，寫下貴組的討論。

   c={5.0, 2.5, 2.25, 2.0, 1.75, 1.50, 1.25, 1.0, 0.5, 0.0}

繳交日期：2008. 3. 26



1
0
-1
u(t) / u(0)
Critically damped, ζ = 1
Overdamped, ζ = 2
Underdamped, ζ = 0.1
1
2
3
t / T_n

複　數　與　簡　諧　振　動

Complex Variable and Harmonic Vibration

序　言

記得 94 年 6 月 9 日在台大土木系碩士班論文口試上，遇見海洋大學陳正宗教授，他是這次論文口試委員，他也是我很早以前台大的學生，他告訴我說：陳老師，你以前發表在機械月刊上〝複數與簡諧振動〞的論文，我把它當作工數及振動學的講義，而且學生學習的效果很好，何不多加流傳，以嘉惠學生。我聽了內心很感動，藉著這次主持〝台大營建知識網〞之便，將這篇論文稍加修正後，刊登在該網站上，希望能獲得大家的認同與喜愛。(該篇文章原刊登在機械月刊第 12 卷第 5 期, May 1986)

前言

本人在台大教授「振動」有關課程多年，一直發現許多學生對以複數表示簡諧振動之基本物理意義不甚了解，甚至覺得奇怪。試著找遍多本有關振動學書籍，卻大多毫無所獲，有時也遇到許多工程師也問到相同的問題，而一般的書籍及論文上又常發現這種複數形式的表示法，會覺得很熟悉，但又說不出其所以然，自然覺得如鯁在喉，有感於斯，本人借這次主編振動與噪音專輯之便，特在此佔一塊小小篇幅介紹這個簡單而且有趣的問題，並說明如何以複數分析簡諧振動之方法。又因簡諧振動乃為一切振動分析之基礎，這也是本人寫這篇文章之另一主要目的。

數學模擬與物理意義

一個單位簡諧力(a unit harmonic force) 可表示為：

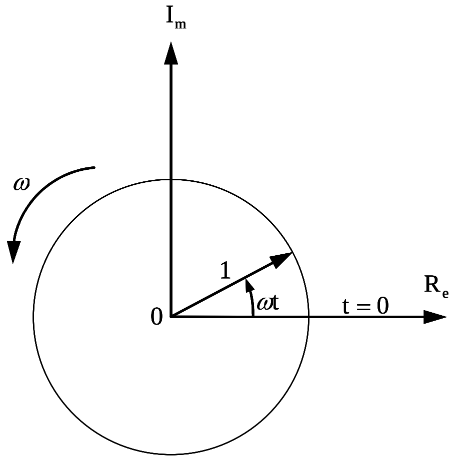


圖1 單位向量$e^{i\omega t}$之相位圖

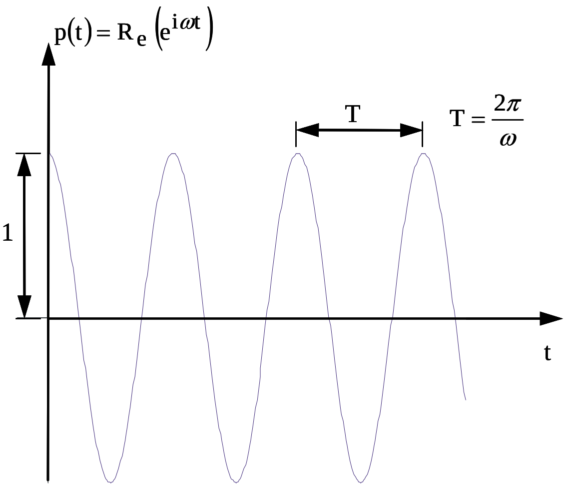


圖2 $p(t)=\cos\omega t$ 之波形

$$p(t)=e^{i\omega t} \cdots\cdots (1)$$

上式中 e 代表指數函數，$i=\sqrt{-1}$，ω 代表頻率(rad/s)，t 代表時間(sec)。

我們必須先了解式(1)僅為外力 $p(t)$之表示法，並不表示 $p(t)$等於$e^{i\omega t}$。因為外力 $p(t)$為一個物理量，是一個時間函數，故為實數，而$e^{i\omega t}=\cos\omega t+i\sin\omega t$ 為一個複數，故無法等於一個實數的物理量。但究竟式(1)代表什麼物理意義？及 $p(t)$與$e^{i\omega t}$之相互關係是什麼？我們利用圖 1 及圖 2 加以說明。

圖 1 係代表一個相位圖，R_e代表實數軸，I_m代表虛數軸。圖上有一個單位向量，當時間$t=0$時，單位向量剛好與R_e重合。此單位向量以ω之轉速作反時鐘方向轉動，故在任何時間 t，此單位向量與R_e軸之夾角為ωt，故圖 1 上之單位向量可表示為$e^{i\omega t}$。圖 2 係表示圖 1 上單位向量在R_e軸上之投影量，故呈弦函數$\cos\omega t$之波形變化，所以如果外力 $p(t)=\cos\omega t$，則在相位圖上可表示為$e^{i\omega t}$，所以 $p(t)$應等於$e^{i\omega t}$之實數部份如下式所示：

$$p(t)=R_e\left(e^{i\omega t}\right)$$

$$=\cos\omega t \cdots\cdots (2)$$

上式中R_e係指括弧( )內之實數部份，所以式(1)右邊$e^{i\omega t}$之實數部份才是真正代表外力 $p(t)$之物理量。

可能有人會問如果$p(t)=\sin\omega t$則又如何以$e^{i\omega t}$來表示？我們可以將此外力 $p(t)$表示為：

$$p(t) = -ie^{i\omega t} \cdots\cdots (3)$$

上式右邊為 $t = 0$ 時，則單位向量剛好在 $-I_m$ 軸上，故此向量作反時鐘方向轉動時，其在 R_e 軸上之投影量即為正弦函數 $\sin \omega t$ 。同樣的式(3)僅是一個簡諧外力之表示法，並不代表 $p(t)$ 等於 $-ie^{i\omega t}$ ，但 $p(t)$ 與 $-ie^{i\omega t}$ 之實數部份應相等，即

$$p(t) = R_e\left(-ie^{i\omega t}\right)$$

$$= \sin \omega t \cdots\cdots (4)$$

所以任一大小及相位之簡諧力可表示為：

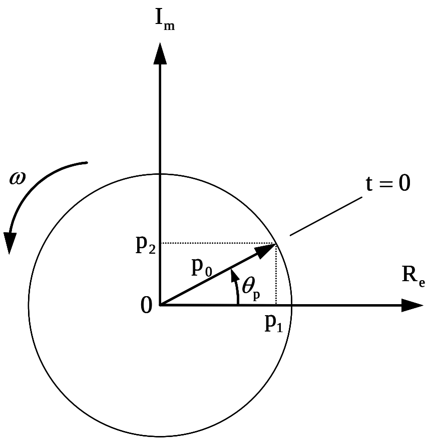
$$p(t) = (p_1 + ip_2)e^{i\omega t} \cdots\cdots (5)$$

上式表示任一簡諧力(不表示相等)，其大小(即振幅) p_0 為 $\sqrt{p_1^{\ 2} + p_2^{\ 2}}$ 及相位 θ_p 如圖 3 所示，如果 p_1 及 p_2 均為正值，則其相位在第一象限內，簡諧力之波形如圖 4 所示，故該簡諧力應等於下式：

$$p(t) = R_e\left[(p_1 + ip_2)e^{i\omega t}\right]$$

$$= p_1 \cos\omega t - p_2 \sin\omega t$$

$$= p_0 \cos(\omega t + \theta_p) \cdots\cdots (6)$$



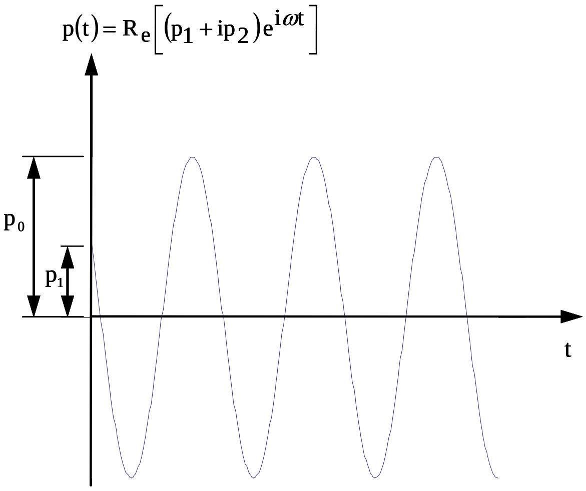


圖3 向量$(p_1+ip_2)e^{i\omega t}$之相位圖　　　　圖4　$p(t)=p_0\cos(\omega t+\theta_p)$之波形

上式中p_0代表$p(t)$之振幅，θ_p代表$p(t)$之相位角如下式所示：

$$p_0=\sqrt{{p_1}^2+{p_2}^2} \cdots\cdots (7)$$

$$\tan\theta_p=\frac{p_2}{p_1} \cdots\cdots (8)$$

所以式(5)右邊係代表一個向量，其大小為$\sqrt{{p_1}^2+{p_2}^2}$，以ω轉速作反時鐘方向轉動。當時間$t=0$時之位置如圖 3 所示，與R_e軸之夾角為θ_p。該向量在R_e軸上之投影量如圖 4 所示，即為簡諧力$p(t)=p_0\cos(\omega t+\theta_p)$之波形。

如果有二個以上之向量(或任意二個簡諧力)，其相位之關係為相對的，所以選擇其中之一向量之相位為參考值(一般假設$\theta=0$)，則其餘向量之相位即可由圖 3 之相位圖很明顯地表示出來。以上就是說明任一簡諧波形以複數表示之基本原理，同樣地簡諧波形亦可以$e^{-i\omega t}$表示之。以下我們來談談如何以複數分析一般簡諧運動，而且可以從分析過程中，看出這種分析方法之精神與優點所在。

簡諧振動之複數分析法

一個一度自由度系統如圖 5 所示，其運動方程式為：

$$m\ddot{y}+c\dot{y}+ky=p(t) \cdots\cdots (9)$$

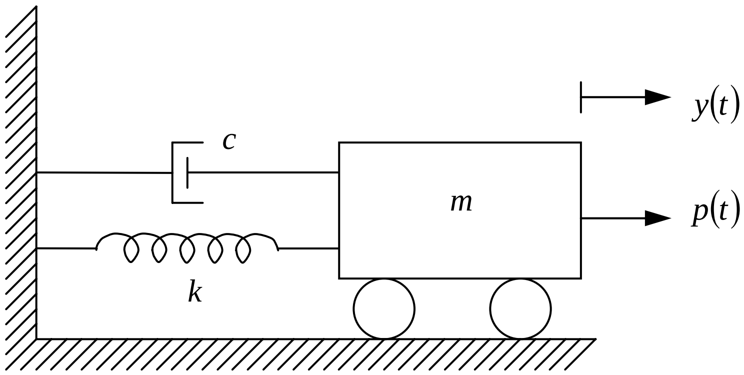


圖5 一度自由度系統

上式中 m, c, k, $p(t)$代表質量、阻尼、勁度及外力；$\ddot{y}$, $\dot{y}$，y 代表加速度、速度及位移。

式(9)二邊同除 m 則變成下式：

$$\ddot{y}+2\xi\omega_n\dot{y}+\omega_n^2 y=\frac{p(t)}{m} \quad (10)$$

上式中ω_n代表自然頻率，ξ代表阻尼比，$\xi=c/2m\omega_n$。

式(10)為一般之一元二次常微分方程式，所以其解為

$$y(t)=y_h(t)+y_p(t) \quad (11)$$

上式中 $y_h(t)$代表齊次解(homogeneous solution)，$y_p(t)$代表特解(particular solution)。因為y_h係與外力$p(t)$無關，而且因阻尼存在會很快衰減，所以當時間 t 增加，y_h之作用就愈不重要，所以又稱為暫態解(transient solution)，因其與外力無關故又稱自由振動解(free-vibration solution)。y_p係與外力$p(t)$之形式有關，如外力作用的時間較為長久，因y_h之影響已不重要，所以$y(t)\doteq y_p(t)$，故y_p又稱為穩態解(steady-state solution)，因其與外力之形式有關，故又稱強制振動解(force-vibration solution)。以下我們來討論如何以複數分析穩態解$y_p(t)$之基本原理。

如果$p(t)$為任一形式之簡諧力，其頻率為ω，故可以如式(5)之複數形式表示之，則穩態解$y_p(t)$亦可以複數表示如：

$$y_p(t)=(y_1+iy_2)e^{i\omega t} \quad (12)$$

也許有人會問為什麼$y_p(t)$可由式(12)表示？又代表些什麼物理意義？一般數學書本上分析一元二次常微分方程式時，如$p(t)$為簡諧形式（$\sin\omega t$ 或 $\cos\omega t$），則即假設$y_p(t)$亦為簡諧形式(即$A\sin\omega t + B\cos\omega t$)。主要係基於將$y_p(t)$之簡諧形式代入常微分方程式中，令方程式二邊相等求解，並沒有提到任何物理意義。但在此我們例舉二個簡單的物理現象來說明，如當一人以ω頻率擊鼓，則我們所聽到的鼓聲頻率也一定是ω。如果係以簡諧形式擊鼓，則聽到的鼓聲一定也是簡諧形式。但擊鼓與聽到鼓聲之時間可能有少許之差異，即兩者之間存在有相位之關係。又如我們聽到一首華爾滋的舞曲，我們也一定以相同之節拍(即頻率)跳華爾滋的舞步，這就是說明輸入(input) $p(t)$與輸出(output) $y_p(t)$具有同步(相同頻率)之物理意義。雖然$p(t)$與$y_p(t)$之頻率相同，但兩者物理量不同，其大小及相位當然就不相同，所以任一簡諧力$p(t)$如式(5)所示，則$y_p(t)$就可以如式(12)表示之。如果系統之阻尼為零，則表示$y_p(t)$與$p(t)$之相位相同，所以阻尼之存在是造成相位差之主要原因，這種現象亦可由以後之式子中導出。

以式(12)代入式(9)之運動方程式中，則左右二邊可同時消去$e^{i\omega t}$，並比較兩邊之實數與虛數部分，可得下列二個方程式：

$$\begin{aligned}(-m\omega^2+k)y_1 - c\omega\, y_2 &= p_1\\ c\omega\, y_1 + (-m\omega^2+k)y_2 &= p_2\end{aligned} \quad \cdots\cdots (13)$$

式(13)可寫成矩陣形式如：

$$\begin{bmatrix}(-m\omega^2+k) & -c\omega\\ c\omega & (-m\omega^2+k)\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\end{Bmatrix} = \begin{Bmatrix}p_1\\ p_2\end{Bmatrix} \quad \cdots\cdots (14)$$

由式(14)可求得y_1及y_2為：

$$\begin{aligned}y_1 &= \frac{(-m\omega^2+k)p_1 + c\omega\, p_2}{(-m\omega^2+k)^2+(c\omega)^2}\\ y_2 &= \frac{-c\omega\, p_1 + (-m\omega^2+k)p_2}{(-m\omega^2+k)^2+(c\omega)^2}\end{aligned} \quad \cdots\cdots (15)$$

因此$y_p(t)$之大小(即振幅)y_0及相位角θ_{yp}則為：

$$y_0 = \sqrt{{y_1}^2 + {y_2}^2} = \frac{p_0}{\sqrt{(-m\omega^2+k)^2+(c\omega)^2}} \quad \cdots\cdots (16)$$

$$\tan\theta_{yp} = \frac{y_2}{y_1} = \frac{-c\omega\, p_1 + \left(-m\omega^2 + k\right)p_2}{\left(-m\omega^2 + k\right)p_1 + c\omega\, p_2} \quad \cdots\cdots (17)$$

所以$p(t)$及$y_p(t)$應等於下列之式子為：

$$p(t) = R_e\left[\left(p_1 + ip_2\right)e^{i\omega t}\right] = p_0 \cos\left(\omega t + \theta_p\right)$$

$$y_p(t) = R_e\left[\left(y_1 + iy_2\right)e^{i\omega t}\right] = y_0 \cos\left(\omega t + \theta_{yp}\right) \quad \cdots\cdots (18)$$

式(15)、(16)、(17)亦可寫成為：

$$y_1 = \frac{\left(1-\beta^2\right)p_1 + \left(2\xi\beta\right)p_2}{m\omega_n^{\,2}\left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]}$$

$$y_2 = \frac{-\left(2\xi\beta\right)p_1 + \left(1-\beta^2\right)p_2}{m\omega_n^{\,2}\left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]} \quad \cdots\cdots (19)$$

$$y_0 = \frac{p_0}{m\omega_n^{\,2}\sqrt{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2}} \quad \cdots\cdots (20)$$

$$\tan\theta_{yp} = \frac{-\left(2\xi\beta\right)p_1 + \left(1-\beta^2\right)p_2}{\left(1-\beta^2\right)p_1 + \left(2\xi\beta\right)p_2} \quad \cdots\cdots (21)$$

上式中ω_n代表自然頻率，$\beta = \omega/\omega_n$。

如果阻尼 c 為零，亦即$\xi = 0$，則$\tan\theta_{yp} = \tan\theta_p$，所以$y_p(t)$與$p(t)$之相位相同。

例題

一個一度自由度系統如圖 5 所示，W, k, c 及$p(t)$等值如下所示，試求(1)位移$y_p(t)$及其與外力$p(t)$之相位關係，及(2)慣性力、阻尼力及彈簧力之向量圖。

$W = 49.6\times10^3$ lb

$k = 100\times10^3$ lb/in

$c = 1125$ lb-s/in

$$p(t)=p_0\sin(\omega t+\theta_p)$$

$$p_0=1000\times10^3\ \text{lb}$$

$$\omega=15\ \text{rad/s}$$

$$\theta=\frac{\pi}{6}$$

解：

（1）先將外力變成複數形式

$$p(t)=p_0\sin(\omega t+30^\circ)=p_0\left(\frac{\sqrt{3}}{2}\sin\omega t+\frac{1}{2}\cos\omega t\right)$$

$$=(p_1+ip_2)e^{i\omega t}$$

所以

$$p_1=\frac{p_0}{2}=500\times10^3\ \text{lb}$$

$$p_2=-\frac{\sqrt{3}}{2}p_0=-866\times10^3\ \text{lb}$$

外力 $p(t)$ 向量係在相位圖第 4 象限上如圖 6 所示

$$m=\frac{W}{g}=\frac{49.6\times10^3}{32.2\times12}=128.4\ \text{lb-s}^2/\text{in}$$

$$\omega_n=\sqrt{\frac{k}{m}}=\sqrt{\frac{100\times10^3}{128.4}}=27.9\ \text{rad/sec}$$

$$\xi=\frac{c}{2m\omega_n}=\frac{1125}{2\times128.4\times27.9}=15.7\%$$

$$\beta=\frac{\omega}{\omega_n}=\frac{15}{27.9}=0.54$$

由式(19)可得 y_1 及 y_2 為：

$$y_1=3.9\ \text{in}$$

$$y_2=-13.1\ \text{in}$$

所以 $y_p(t)$ 之振幅及相位為：

$$y_0=\sqrt{y_1^{\ 2}+y_2^{\ 2}}=13.7\ \text{in}$$

$\tan\theta_{yp} = \dfrac{y_2}{y_1} = -3.36$（第 4 象限）

故 $y_p(t)$ 之波形為：

$$y_p(t) = 13.7\sin\left(\omega t + 16.57^\circ\right) \text{in}$$

圖 6 及圖 7 分別表示外力 $p(t)$ 及反應 $y_p(t)$ 之相位圖，所以 $y_p(t)$ 落後 $p(t)$ 之角度為：

$$\Delta\theta = 73.43^\circ - 60^\circ = 13.43^\circ$$

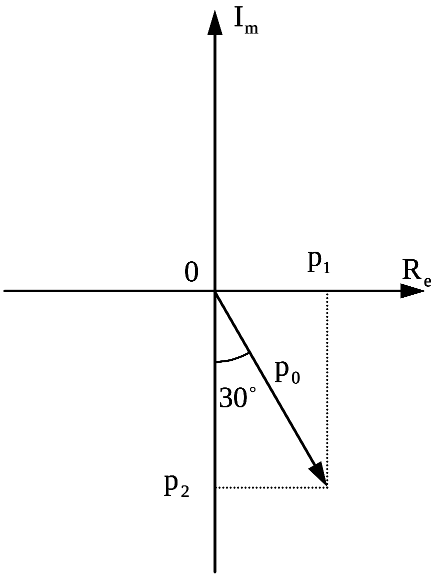


圖6 外力p(t) 之相位圖

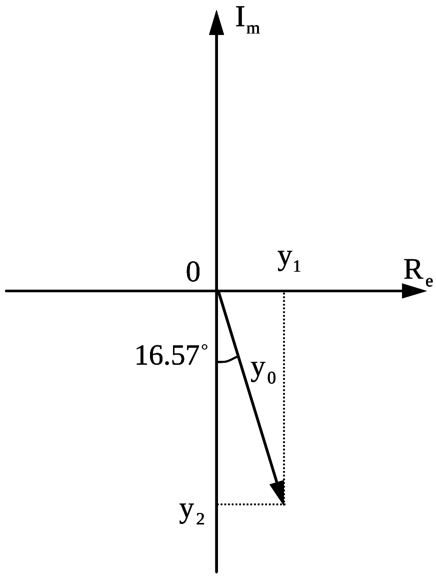


圖7 位移 $y_p(t)$ 之相位圖

（2）慣性力 $f_i(t) = m\ddot{y}_p(t) = -m\omega^2\, y_p(t)$

所以慣性力 $f_i(t)$ 與 $y_p(t)$ 反向，即相位差 180°，而慣性力之大小(或稱振幅 amplitude）F_i 如下：

$$F_i = m\omega^2 y_0$$

$$= 128 \times (15)^2 \times 13.7 = 395.79 \times 10^3 \text{ lb}$$

阻尼力 $f_d(t) = c\dot{y}_p(t) = -i\, c\omega\, y_p(t)$

所以阻尼力 $f_d(t)$ 與 $y_p(t)$ 相位差 90°，$f_d(t)$ 超前 $y_p(t)$ 90°，如圖 8 所示，

阻尼力之大小 F_d 如下：

$$F_d = c\omega\, y_0$$

$$= 1125 \times 15 \times 13.7 = 231.19 \times 10^3 \text{ lb}$$

彈簧力 $f_s(t) = k\, y_p(t)$

所以彈簧力 $f_s(t)$ 與 $y_p(t)$ 同向(相位同)，其大小如下：

$$F_s = k y_0$$

$$= 100 \times 10^3 \times 13.7 = 1370 \times 10^3 \text{ lb}$$

慣性力、阻尼力、及彈簧力之向量如圖 8 所示，以上三者合向量應與 $p(t)$ 同(包括大小及相位)。

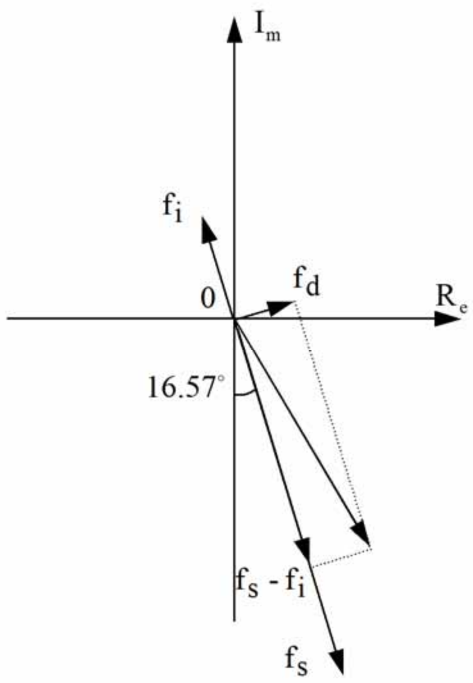


圖8 f_i, f_d 及 f_s 之相位圖

自由振動(Free Vibration)

如圖 5 所示之一度自由度系統做自由振動，即表示圖 5 中 $p(t) = 0$，或式(14)中 $p_1 = p_2 = 0$，如式(14)有根存在，則式(14)左邊矩陣之行列式必須等於零，故得

$$\left(-m\omega^2 + k\right)^2 + (c\omega)^2 = 0 \cdots\cdots (22)$$

上式即為圖 5 所示之一度自由度系統之頻率方程式(frequency equation)。

式(22)可簡化變成

$$m\omega^2 \pm ic\omega - k = 0 \cdots\cdots (23)$$

或改寫為

$$\omega^2 \pm 2i\xi\omega_n\omega - \omega_n{}^2 = 0 \cdots\cdots (24)$$

上式中ξ及ω_n分別代表該系統之阻尼比及自然頻率。
式(24)為二次代數方程式，其解ω如下式所示：

$$\omega = \pm i\xi\,\omega_n \pm \sqrt{1-\xi^2}\,\omega_n \cdots\cdots (25)$$

因為一般系統之阻尼比遠小於 1.0(即ξ<1.0)，故式(25)可寫成

$$\omega = \pm i\xi\omega_n \pm \omega_d \cdots\cdots (26)$$

上式中ω_d為系統之阻尼頻率(damped frequency)，如下式

$$\omega_d = \sqrt{1-\xi^2}\,\omega_n \cdots\cdots (27)$$

該系統之自由振動解(即齊次解)$y_h(t)$亦可由式(12)表示之，即

$$y_h(t) = (y_1 + iy_2)e^{i\omega t} \cdots\cdots (28)$$

取式(28)右邊之實數部份即為$y_h(t)$之物理量，故

$$y_h(t) = R_e(y_1 + iy_2)e^{i\omega t} \cdots\cdots (29)$$

為不使$y_h(t)$發散，所以式(26)右邊第一項應取正值，故

$$\omega = i\xi\omega_n \pm \omega_d \cdots\cdots (30)$$

將式(30)代入式(29)得$y_h(t)$之解為

$$y_h(t) = e^{-\xi\omega_n t}\,(A\sin\omega_d t + B\cos\omega_d t) \cdots\cdots (31)$$

式(31)常數 A 及 B 可由系統之初始條件決定之。
如該系統之阻尼比為零，為一無阻尼系統(undamped system)，即$\xi = 0$，及

$\omega_d = \omega_n$ ，所以式(31)變成

$$y_h(t) = A\sin\omega_n t + B\cos\omega_n t \cdots\cdots (32)$$

Complex variable for ODE(一竿子打兩個)

Real×1 + Imaginary×i

$$m\ddot{v}(t)+c\dot{v}(t)+kv(t)=P_0\cos\varpi t \rightarrow m\ddot{z}(t)+c\dot{z}(t)+kz(t)=P_0e^{i\varpi t} \leftarrow m\ddot{v}(t)+c\dot{v}(t)+kv(t)=P_0\sin\varpi t$$

$$v_p(t)=P_R\cos\varpi t+Q_R\sin\varpi t \qquad z(t)=Ze^{i\varpi t} \qquad v_p(t)=P_I\cos\varpi t+Q_I\sin\varpi t$$

$$P_R=\frac{P_0}{k}\frac{(1-\beta^2)}{(1-\beta^2)^2+(2\xi\beta)^2}$$

$$Q_R=\frac{P_0}{k}\frac{(2\xi\beta)}{(1-\beta^2)^2+(2\xi\beta)^2}$$

$$\Rightarrow \text{ where } \beta=\varpi\sqrt{\frac{m}{k}}=\varpi/\omega_n$$

$$\xi=\frac{c}{2}\sqrt{\frac{1}{km}}=\frac{c}{2m\omega_n}$$

$$Z=z_R+iz_I$$

$$Z=\frac{P_0}{k}\frac{(1-\beta^2)}{(1-\beta^2)^2+(2\xi\beta)^2}-i\frac{P_0}{k}\frac{(2\xi\beta)}{(1-\beta^2)^2+(2\xi\beta)^2}$$

$$P_I=-\frac{P_0}{k}\frac{(2\xi\beta)}{(1-\beta^2)^2+(2\xi\beta)^2}$$

$$Q_I=\frac{P_0}{k}\frac{(1-\beta^2)}{(1-\beta^2)^2+(2\xi\beta)^2}$$

$$z(t)=\frac{P_0}{k}\frac{(1-\beta^2)}{(1-\beta^2)^2+(2\xi\beta)^2}[\cos\varpi t+i\sin\varpi t]-i\frac{P_0}{k}\frac{(2\xi\beta)}{(1-\beta^2)^2+(2\xi\beta)^2}[\cos\varpi t+i\sin\varpi t]$$

Complex variable for ODE

Real×1 + Imaginary×i

Real×1

$$\rho = \frac{P_0}{k}\left(\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right)^{\frac{-1}{2}}$$

$$v_p(t) = \rho\cos(\varpi t + \theta)$$

$$\theta = \tan^{-1} - \frac{Q_R}{P_R} = \tan^{-1}\left(-\frac{2\xi\beta}{1-\beta^2}\right)$$

+

$$Z = P_R - iQ_R = \rho e^{i\theta}$$

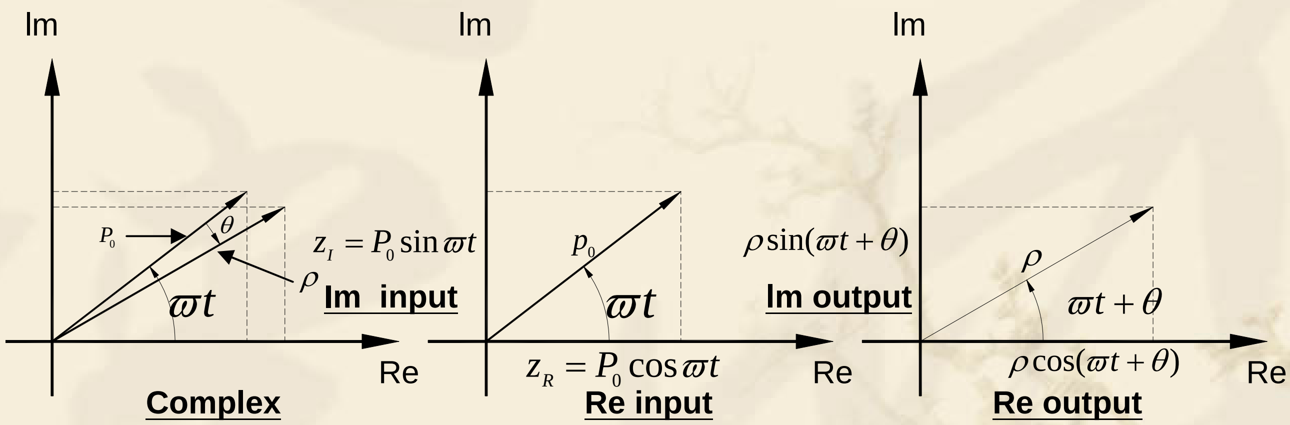
$$z(t) = \rho e^{i\theta} e^{i\varpi t} = \rho e^{i(\theta + \varpi t)}$$

Imaginary×i

$$\rho = \frac{P_0}{k}\left(\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right)^{\frac{-1}{2}}$$

$$v_p(t) = \rho\sin(\varpi t + \theta)$$

$$\theta = \tan^{-1} - \frac{Q_I}{P_I} = \tan^{-1}\left(-\frac{2\xi\beta}{1-\beta^2}\right)$$



M=1; ξ =0.1; k=1 ; θ =-7.5946°

Two sources: external excitation or free vibration with two near frequencies

External excitation

$$\ddot{x}(t)+\omega^2 x(t)=F cos(\Omega t)$$

General solution

$$x(t)=c_1 cos(\omega t)+c_2 sin(\omega t)+\frac{F}{(\omega^2-\Omega^2)}cos(\Omega t)$$

Initial conditions to determine c_1 and c_2

$$x(0)=0, \dot{x}(0)=0$$

Total solution

$$x(t)=\frac{F}{(\omega^2-\Omega^2)}\{cos(\Omega t)-cos(\omega t)\}$$

Stage 1: excitation

$$x(t)=\frac{F}{(\omega^2-\Omega^2)}\{cos(\Omega t)-cos(\omega t)\}$$

Stage 2: beating ($\omega \doteq \Omega$)

$$x(t)=\frac{F}{(\omega^2-\Omega^2)}\{cos(\Omega t)-cos(\omega t)\}$$

$$x(t)=\frac{-2F}{(\omega^2-\Omega^2)}sin(\frac{\Omega+\omega}{2})t\, sin(\frac{\Omega-\omega}{2})t \rightarrow beating$$

$$x(t)=\frac{2F}{(2\omega)(2\epsilon)}sin(\omega t)\, sin(\epsilon t)$$

$$x(t)=\frac{Ft}{2\omega}sin(\omega t)\ \rightarrow resonance$$

where $\Omega-\omega=2\epsilon$.

Stage 3: resonance ($\omega=\Omega$)

$$x(t)=\frac{F}{(\omega^2-\Omega^2)}\{cos(\Omega t)-cos(\omega t)\}\rightarrow\infty$$

$$x(t)=\frac{Ft}{2\omega}sin(\omega t)\ \rightarrow\infty, as\ t\rightarrow\infty$$

Two sources: external excitation or free vibration with two near frequencies

case 2: free vibration with two near natural frequencies (pendulum)

$$\omega_1 = \sqrt{\frac{g}{L}}$$

$$\omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}\frac{a^2}{L^2}}$$

Initial conditions to determine coefficients

$$\theta_1(0) = \theta_0, \dot{\theta}_1(0) = 0$$

$$\theta_2(0) = 0, \dot{\theta}_2(0) = 0$$

General solution

$$\theta_1(t) = \frac{1}{2}\theta_0 cos(\omega_1 t) + \frac{1}{2}\theta_0 cos(\omega_2 t)$$

$$\theta_2(t) = \frac{1}{2}\theta_0 cos(\omega_1 t) - \frac{1}{2}\theta_0 cos(\omega_2 t)$$

General solution for beating as $\frac{ka^2}{mL^2}$ is very small

$$\theta_1(t) = \theta_0 cos(\frac{\omega_2 - \omega_1}{2})t \; cos(\frac{\omega_2 + \omega_1}{2})t$$

$$\theta_2(t) = \theta_0 sin(\frac{\omega_2 - \omega_1}{2})t \; sin(\frac{\omega_2 + \omega_1}{2})t$$

As $\frac{ka^2}{mL^2} = 0$, reduce to simple pendulum.

$$\theta_1(t) = \theta_0 cos(\omega_1 t)$$

$$\theta_2(t) = 0$$

Excitation case by support motion instead of external excitation

Two sources: external excitation or free vibration with two near frequencies

External excitation

$$\ddot{x}(t) + \omega^2 x(t) = Fcos(\omega t)$$

By variation of parameters:

$$x(t) = u_1 cos(\omega t) + u_2 sin(\omega t)$$

Two constraints:

$$cos(\omega t)u_1' + sin(\omega t)u_2' = 0$$

$$-\omega sin(\omega t)u_1' + \omega cos(\omega t)u_2' = Fcos(\omega t)$$

Solve u_1', u_2'

$$u_1'(t) = \frac{-F}{2\omega} sin(2\omega t)$$

$$u_2'(t) = \frac{F}{2\omega}(1 + cos(2\omega t))$$

Solve u_1, u_2

$$u_1(t) = \frac{F}{4\omega^2} cos(2\omega t) + c_1$$

$$u_2(t) = \frac{F}{2\omega}(t + \frac{1}{2\omega} sin(2\omega t)) + c_2$$

Particular solution contains a complementary solution:

$$x(t) = \frac{Ft}{2\omega} sin(\omega t) + \frac{F}{4\omega^2} cos(\omega t)$$

Particular solution:

$$x(t) = \frac{Ft}{2\omega} sin(\omega t) \ \rightarrow resonance$$

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教學中心

網路輔助教學經驗分享(複變函數)

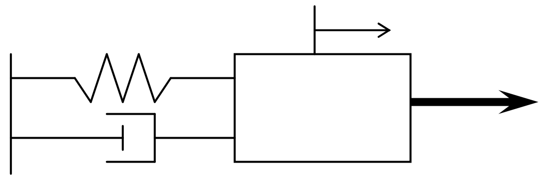
J T Chen (陳正宗特聘教授)

河工系

14:55-15:00, Dec. 13, 2006

(網路教學2007-ode.doc)

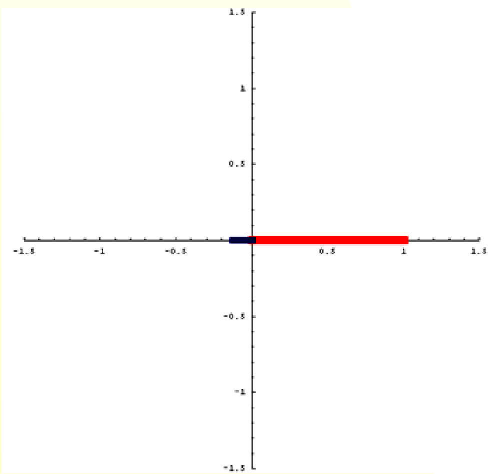
輸出輸入振幅比與相角差-動畫



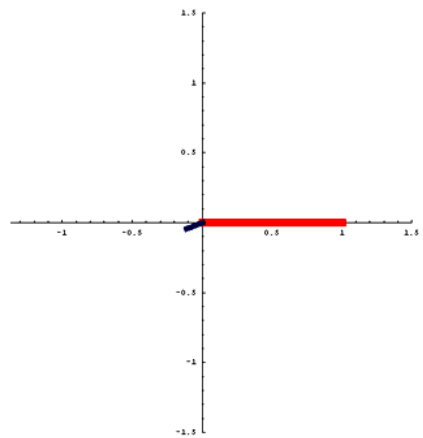
output

input

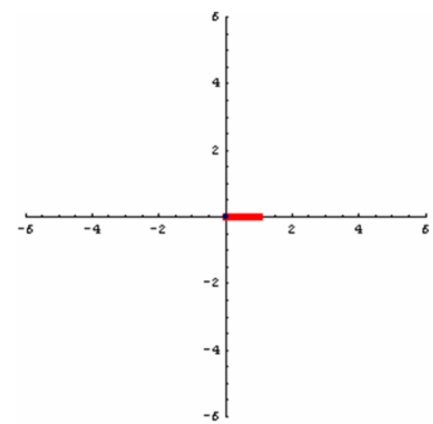
No Damping



Damping



Resonance



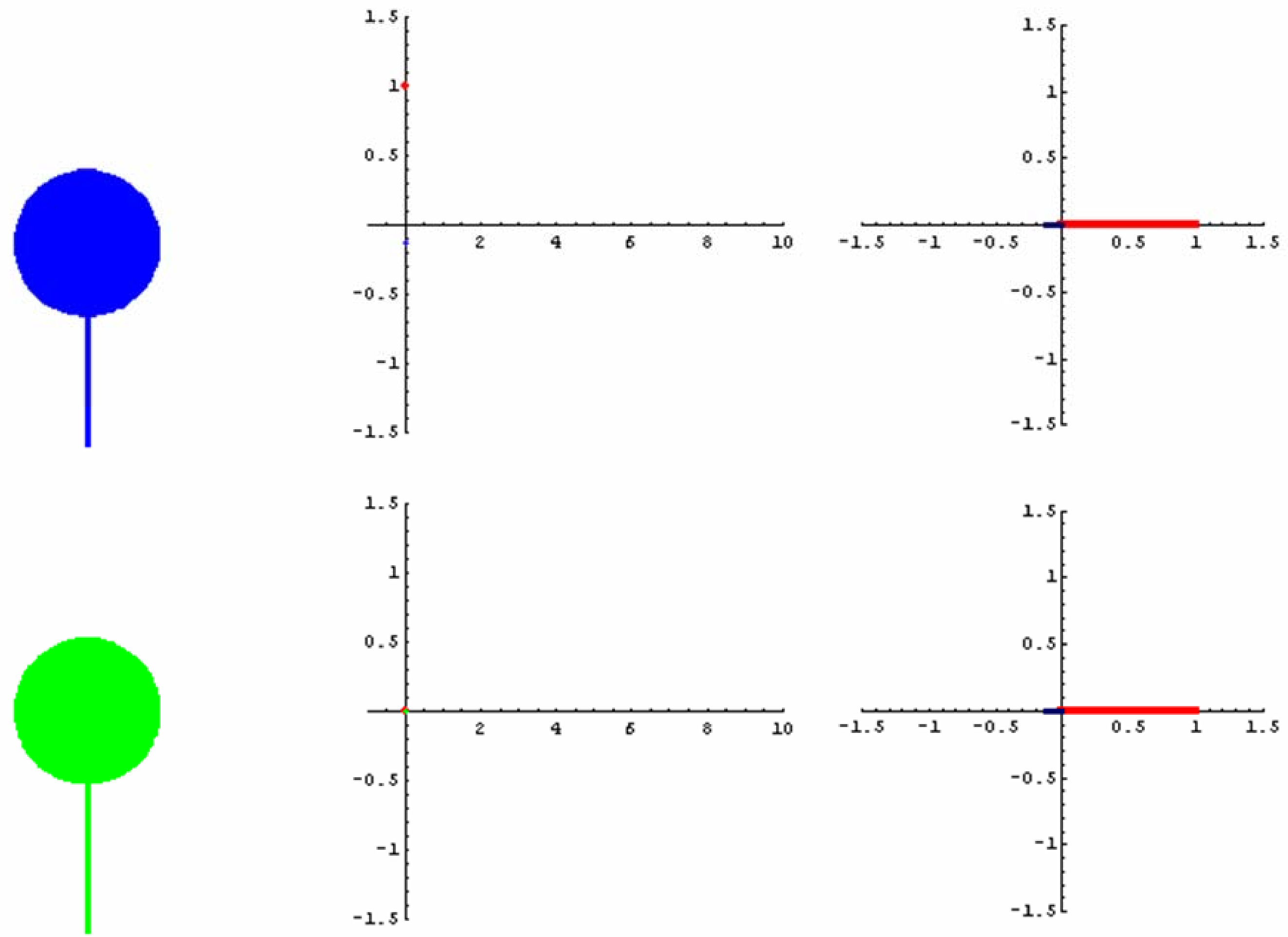
陳永祥,複數與簡諧振動,機械月刊第十五期, pp.76-79, 1986.

output

input

No Damping

Real



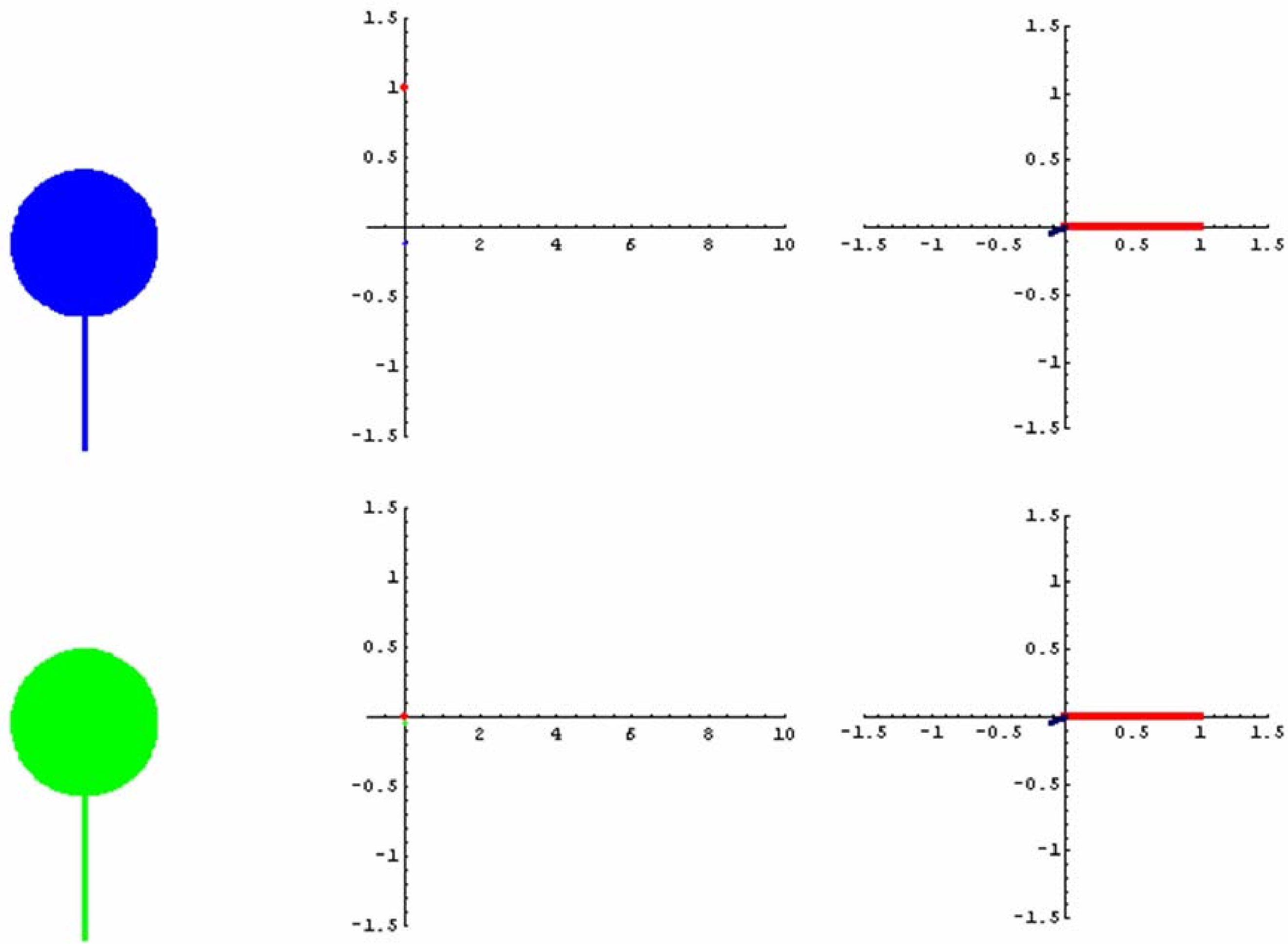
Imaginary

—— output

—— input

Damping

Real

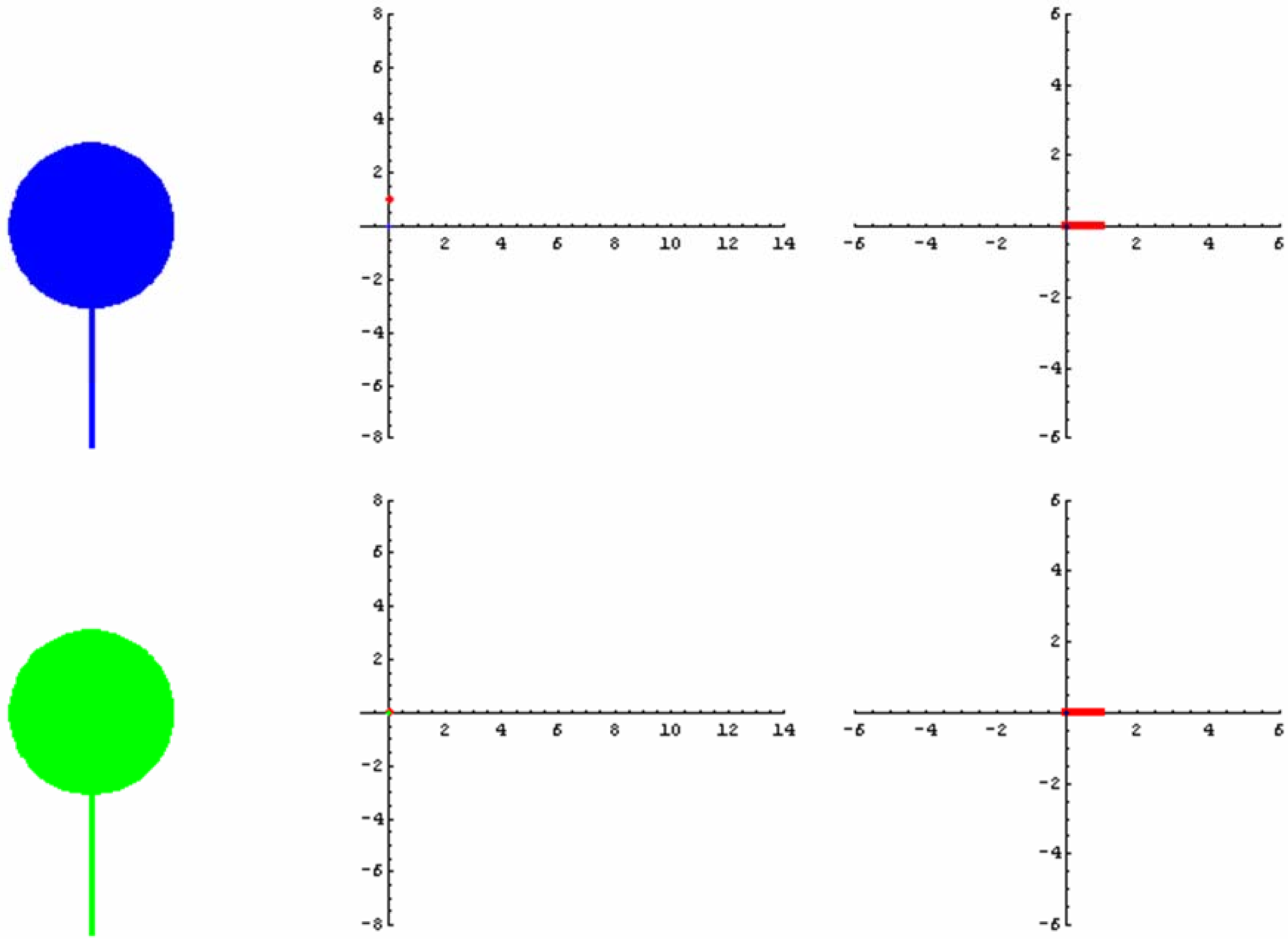


Imaginary

—— output
—— input

Resonance

Real



Imaginary

歡迎參觀海洋大學力學聲響振動實驗室
烘焙雞及捎來伊妹兒

http://ind.ntou.edu.tw/~msvlab/

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Papers

如有任何意見請 E-mail 告訴我們，本網頁負責人：陳正宗，李應德，廖奐禎，柯佳男，方衛尉，周克勳

網頁內容部分檔案格式為*.PDF，請安裝Acrobat Reader來瀏覽

NEW	06/11/08	ICOME 2006 邀陳正宗 特聘教授 Plenary lecture
NEW	06/11/07	邊界元素法入門一書，近期推出，欲購者請洽本研究室或留言告知
NEW	06/11/06	恭賀吳國綸學弟甄試高中台大土木工程學系研究所-水利組、結構組碩士班
NEW	06/11/02	下星期一(11/6)晚上7點~9點工數(一)期中考，在河二館404與405教室，請攜帶學生證以備查驗!
NEW	06/11/01	本研究室研究成果已突破410篇論文引用
NEW	06/11/01	恭賀柯永彥博士進入國家地震中心服務
NEW	06/10/30	恭賀吳國綸,蔡明宏,林裕棨獲國立臺灣海洋大學94學年度第2學期書卷獎
NEW	06/10/27	恭賀吳安傑,陳柏源,高政宏,蕭嘉俊學長獲國立臺灣海洋大學95年度「大學部及碩士班在校生論文發表於SC
NEW	06/10/25	恭賀河工系陳正宗、周宗仁、陳俶季、黃然、蕭葆羲與簡連貴等教授在校服務期間得免辦理評估
NEW	06/10/24	恭賀 吳安傑學長 入圍第三十屆力學會議學生論文競賽
NEW	06/10/13	新加坡南洋科技大學提供博士生獎學金一名(從事邊界元素法研究)
NEW	06/10/13	恭賀林盛益學長普考土木科通過
NEW	06/10/11	感謝中興工程科技研究基金會再度贊助李應德學長博士班獎助學金
NEW	06/09/27	對偶邊界元素法世界最具影響力學者前三名: Aliabadi M.H. , Chen J.T. , Power H.
NEW	06/09/27	邊界元素法世界最具影響力學者前四名:Aliabadi M.H. ,Mukherjee.S , Chen J.T. , Tanaka M.

Frame indifference: objectivity

Problem: a SDOF mass-spring system with a mass m and a spring constant k in equilibrium. A half mass drops out suddenly, Please describe the vibration phenomenon.

Reference 1: $x_1(t)$ positive downward from original length of spring

$$\textit{governing equation}\ \frac{m}{2}\ddot{x}_1(t)+kx_1(t)=\frac{mg}{2}$$

$$\textit{initial conditions}\ x_1(0)=mg/k, \dot{x}_1(0)=0$$

$$\textit{solution}\ x_1(t)=\frac{mg}{2k}cos(\omega_1 t)+\frac{mg}{2k}>0, for\ any\ t$$

where $\omega_1^2 = 2k/m$.

Reference 2: $x_2(t)$ positive downward from equilibrium point

$$\textit{governing equation}\ \frac{m}{2}\ddot{x}_2(t)+k(x_2(t)+mg/k)=\frac{mg}{2}$$

$$\textit{initial conditions}\ x_2(0)=mg/k, \dot{x}_2(0)=0$$

$$\textit{solution}\ x_2(t)=\frac{mg}{2k}cos(\omega_1 t)-\frac{mg}{2k}$$

Reference 3: $x_3(t)$ positive downward from the lowerest point

$$\textit{governing equation}\ \frac{m}{2}\ddot{x}_3(t)+k(x_2(t)+2mg/k)=\frac{mg}{2}$$

$$\textit{initial conditions}\ x_3(0)=-mg/k, \dot{x}_3(0)=0$$

$$\textit{solution}\ x_3(t)=\frac{mg}{2k}cos(\omega_1 t)-\frac{3mg}{2k}$$

Discussions:

$$x_1(t)=x_2(t)+\frac{mg}{k}=x_3(t)+\frac{2mg}{k}$$

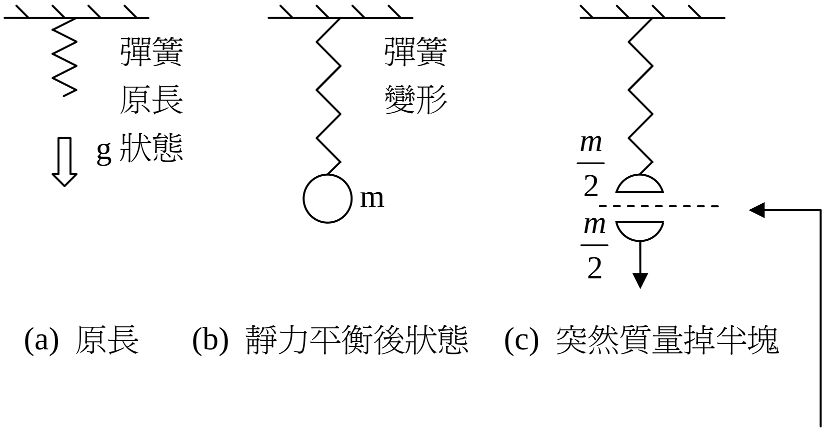
All the three solutions are the same in the physical sense.

All the three solutions obeys the objectivity, i.e. frame indifference

Please plot the three solutions by mathematica.

筆試

一質點質量 m 慢慢徐徐掛上彈簧常數 k 的彈簧後(考慮質點自重 mg)，後來質點掉下一半質量，請定性與定量描述整個動力反應(座標原點可自定)並繪出最低點與最高點及振動周期。(Hint:建數學模式) 參見下圖:



(a) 原長　(b) 靜力平衡後狀態　(c) 突然質量掉半塊

(Hint:座標原點隨您設)　　　接下來上半部$\frac{m}{2}$質點會怎麼動？

振動三部曲-激發、拍擊與共振(using Laplace transform)

海大河海系 陳正宗

Two sources: external excitation or free vibration with two near frequencies

External excitation

$$\ddot{x}(t) + \omega^2 x(t) = cos(\omega t)$$

Initial conditions:

$$x(0) = 0, \dot{x}(0) = 0$$

By taking Laplace transform, we have

$$(s^2 + \omega^2)X(s) = \frac{s}{s^2 + \omega^2}$$

Method 1: using $\mathcal{L}\{tf(t)\} = -F'(s)$

$$X(s) = \frac{s}{(s^2 + \omega^2)^2}$$

$$X(s) = \frac{-1}{2}\frac{-2s}{(s^2 + \omega^2)^2}$$

$$X(s) = \frac{1}{2\omega}(-1)\frac{d}{ds}\{\frac{\omega}{(s^2 + \omega^2)}\}$$

Method 2: using convolution

$$X(s) = \frac{1}{(s^2 + \omega^2)}\frac{s}{(s^2 + \omega^2)}$$

$$x(t) = \int_0^t \frac{sin(\omega(t-\tau))}{\omega} cos(\omega\tau) d\tau$$

Total solution:

$$x(t) = \frac{t}{2\omega} sin(\omega t) \rightarrow resonance$$

Bessel function	Lengndre function
$J_\nu(x)$	$P_n(x)$
membrane vibration	Gaussian integration, spherical harmonic
$x^2y" + xy' + (x^2-\nu^2)y = 0$	$(1-x^2)y" - 2xy' + (n)(n+1)y = 0$
$\frac{d}{dx}(xy') + (x - \frac{\nu^2}{x})y = 0$	$\frac{d}{dx}\{(1-x^2)y'\} + n(n+1)y = 0$
$x = 0$ regular singular	$x = 0$ ordinary point
$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$	$y(x) = \sum_{n=0}^{\infty} c_n x^n$
$J_\nu, J_{-\nu}, J_n, Y_n, J_0, Y_0$	$P_n(x), n \in N, n\, not \in N$
$J_p(x) = \mid x/2 \mid^p \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n}}{n!\Gamma(p+n+1)}$	$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\{(x^2-1)^n\}$
$\int_0^\infty x J_p(x) J_q(x) dx = 0$	$\int_{-1}^{1} P_i(x) P_j(x) dx = 0$

Mechanical system (m, c, k)

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = p(t)$$

Electrical system (L, R, C)

$$L\ddot{I}(t) + R\dot{I}(t) + CI(t) = E(t)$$

工程數學

求補解的三種方法：

一、尤拉式：

$x^2\, y'' + a\, x\, y' + b\, y = 0$

(1) $y = x^m$

$m(m-1) + a\,m + b = 0$

$y = x^{m_1}$ 、 $y = x^{m_2}$

(2) y_1已知， $y_2 = y_1 u_1$

$y'' + \frac{a}{x} y' + \frac{b}{x^2} y = 0$

欲求 u_1 與 y_2

設：$u' = v$ 代回上式：

$y_1 v' + (2y_1' + \frac{a}{x} y_1) v = 0$

求得： v

$u_1 = \int v\, dx \qquad y_2 = y_1 u_1$

(3) $W(x) = e^{-\int (\frac{a}{x})\, dx}$

$y_1\, y_2' - y_1'\, y_2 = e^{-\int \frac{a}{x} dx}$

二、一般式：

$y'' + p(x)\, y' + q\, y = 0$

(2) y_1已知， $y_2 = y_1 u_1$

$y_1\, u_1'' + [\,2\, y_1' + p(x)\, y_1\,]\, u_1' = 0$ 成立

欲求： u_1 與 y_2

設：$u' = v$ 代回上式：

$y_1\, v' + [\; 2\, y_1' + p(x)\, y_1\,]\, v = 0$

求得： v

$u_1 = \int v\, dx \qquad y_2 = y_1 u_1$

(3) $W(x) = e^{-\int p(x) dx}$

$y_1\, y_2' - y_1'\, y_2 = e^{-\int p(x) dx}$

[註]： $[\, p(x) = a/x$ 、$q(x) = b/x^2\,]$

[註]：一、尤拉式（1）（2）（3）皆可用　二、一般式（2）（3）可用

If $y = x$ **is one complementary solution of** $(1-x^2)y'' - 2xy' + 2y = 0$**, find the other one. (1)** $y_2 = y_1 u_1$ **(2)Wronskian (3)** $\left(\frac{y_2}{y_1}\right)'$

(1) $y_2 = y_1 u$

$$(1-x^2)xu'' + (2-4x^2)u' = 0$$

$$u' = v$$

$$(1-x^2)xv' + (2-4x^2)v = 0$$

$$v = \frac{k}{x^2(1-x^2)}$$

$$u = \int v dx = \frac{-1}{x} + \ln\sqrt{\frac{1+x}{1-x}} + c$$

$$y_2 = 1 + x\ln\sqrt{\frac{1+x}{1-x}}$$

(2) *Wronskain*

$$w' - \frac{2x}{1-x^2}w = 0 \rightarrow w(x) = \frac{1}{1-x^2}$$

$$w = y_2 y_1' - y_1 y_2' = \frac{1}{1-x^2}$$

$$-xy_2' + y_2 = \frac{1}{1-x^2}$$

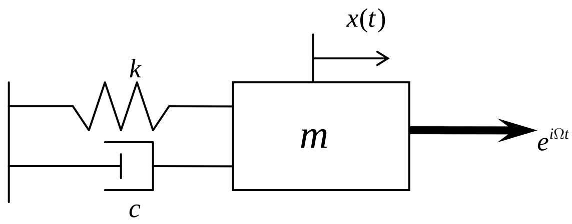
$$y_2 = 1 + x\ln\sqrt{\frac{1+x}{1-x}}$$

(3) $$\left(\frac{y_2}{y_1}\right)' = \frac{-y_2 y_1' + y_1 y_2'}{y_1^2} = \frac{-w}{y_1^2}$$

$$y_2 = y_1 \int \frac{1}{y_1^2} w dx$$

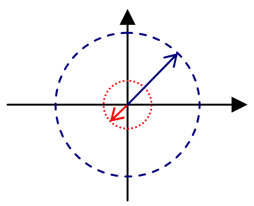
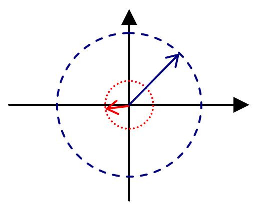
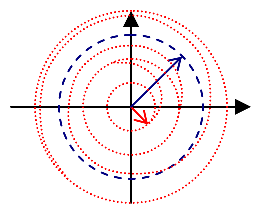
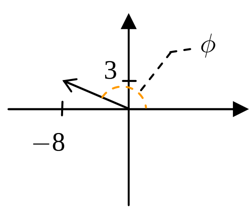
$$y_2 = 1 + x\ln\sqrt{\frac{1+x}{1-x}}$$

Forced vibration of m,c and k system



$$m\ddot{x}+c\dot{x}+kx=e^{i\Omega t} \quad \text{m=1,k=1} \quad \Omega=3$$

Phase lag between [input ••••••] and [output ••••••]

No Damping	Damping	No damping & renonance
$(c=0)$	$(c\neq 0)$, c=1	$\Omega=\omega=\sqrt{\frac{k}{m}}$ $(c=0)$
		
$\phi=0,\pi,2\pi...$	$\phi\neq 0,\pi,2\pi...$	$\phi=\frac{3}{2}\pi$
Amplitude change：$\frac{1}{8}$ Phase lag：$\phi=\pi$	Amplitude change：$\frac{1}{\sqrt{73}}$ Phase lag： 	Amplitude change：$\frac{t}{2\omega}$ Phase lag：$\phi=\frac{3}{2}\pi$

第三章 矩陣(含張量)

向量與線性代數基本能力測試 2B(2008)

$\underset{\sim}{a} = (-1,4,1)$, $\underset{\sim}{b} = (0,1,0)$, $\underset{\sim}{c} = (1,0,0)$ 求

1. $|a| =$ ________

2. $\underset{\sim}{a} \cdot \underset{\sim}{b} =$ ________

3. $\underset{\sim}{a} \times \underset{\sim}{b} =$ ________

4. $\underset{\sim}{a} \times \underset{\sim}{b} \cdot \underset{\sim}{c} =$ ________

5. $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ are linear independent？ (Yes or No) ________

6. $\det\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} =$ ________

7. $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} =$ ________

8. $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} =$ ________

9. Rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix} =$ ________

10. $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ is singular？ (Yes or No) ________

11. $A\underset{\sim}{x} = 0$, Find $\underset{\sim}{x}$ $(\underset{\sim}{x} \neq 0)$ _______

12. $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} v = \lambda v$, find eigenvectors and eigenvalues.

- Vector :

  1. directional cosines
  2. inner product(work,projection)
  3. outer product(area vector,moment)
  4. unit vecor
  5. maginitude, length
  6. point description
  7. projection theorem
  8. triple product
  9. line description
  10. plane description

- Vector space:

  1. linear combinations
  2. subspace
  3. basis
  4. dimension
  5. linearly dependent
  6. linearlly independent
  7. Gram-Schimidt process
  8. orthogonal
  9. row vector, column vector
  10. functional space

- Tensor :

  1. rank 0 — scalar(length, enengy, work)
  2. rank 1 — vector(position,displacement, velocity, acceleration)
  3. rank 2 — moment of inertia, stress and strain
  4. transformation law

張量(tensor)與其階數：

滿足某種轉換關係的物理量稱之為張量。(Transformation law)

以下利用不同座標系來說明一階張量（向量）的轉換關係。

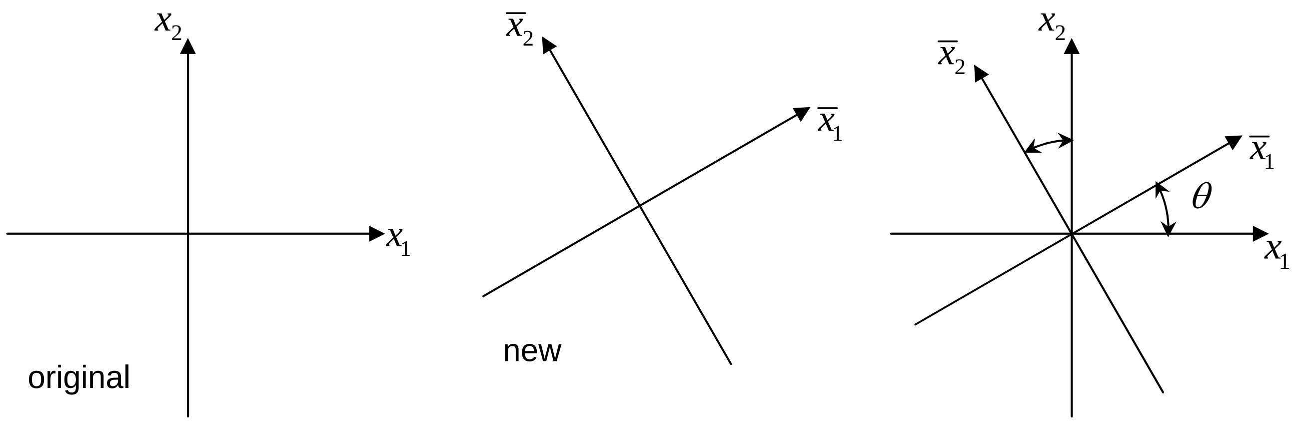


圖 1-a　　圖 1-b　　圖 1-c

想法：

同一物理量，不同觀測者利用不同座標系統觀測時，所觀測結果應為等價的，故座標間存在某種轉換關係使其相等。

$$\begin{Bmatrix} \bar{x}_1 \\ \bar{y}_2 \end{Bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{Bmatrix} x_1 \\ y_2 \end{Bmatrix} \tag{1}$$

階數則以轉換次數來看，零階張量為純量，無須轉換，而向量（一階張量）則需一次（如（1）式），慣量等二階張量則需要 2 次的轉換。其中 $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ 為轉換矩陣。

轉換矩陣(Transformation matrix)：

新舊座標間的轉換矩陣，可利用新舊軸間的餘弦關係表示，說明如下。

若轉換矩陣為 $[L]$，其內元素以 L_{ij} 表示，而 θ_{ij} 為新 i 軸與舊 j 軸的夾角，則：

$$L_{ij} = \cos(\theta_{ij}) \tag{2}$$

故可得轉換矩陣為

$$\begin{bmatrix} \cos(\theta_{\bar{x}x}) & \cos(\theta_{\bar{x}y}) \\ \cos(\theta_{\bar{y}x}) & \cos(\theta_{\bar{y}y}) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{3}$$

※此種轉換關係與高中所學之座標轉換單元相同，可參見高中筆記或講義。

愛因斯坦符號(Einstein symbol)：

一種級數的表現方式，利用一般項來表示級數。如：

$$\begin{cases} \overline{x}_1 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 = \sum_{i=1}^{3} a_{i1}x_i \\ \overline{x}_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 = \sum_{i=1}^{3} a_{i2}x_i \\ \overline{x}_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 = \sum_{i=1}^{3} a_{i3}x_i \end{cases} \rightarrow \overline{x}_j = \sum_{i=1}^{3} a_{ij}x_i \rightarrow \overline{x}_j = a_{ij}x_i$$

$\overline{x}_j = a_{ij}x_i$ 是利用「愛因斯坦符號」所寫的級數運算式，其中 j（未重複）為第 j 項之意，而 i(重複)代表級數需代換加總之部分。

(1) 向量內積：

$\underset{\sim}{\boldsymbol{u}} = \langle u_1, u_2, u_3 \rangle$ 與 $\underset{\sim}{\boldsymbol{v}} = \langle v_1, v_2, v_3 \rangle$ 做內積，依據定義為：

$$\underset{\sim}{\boldsymbol{u}} \cdot \underset{\sim}{\boldsymbol{v}} = u_1v_1 + u_2v_2 + u_3v_3$$

利用愛因斯坦符號表示即為：

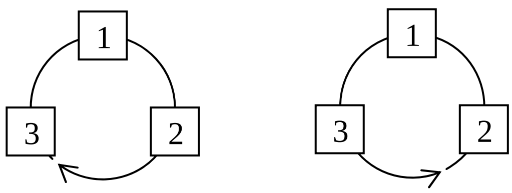
$$\underset{\sim}{\boldsymbol{u}} \cdot \underset{\sim}{\boldsymbol{v}} = u_iv_i = c$$

(2) 輪換張量(permutation tensor, ε_{ijk})：

遵守右手定則之空間座標系統，其單位向量依序為 e_1 、 e_2 與 e_3 ，則輪換張量定義為：

$$\varepsilon_{ijk} = e_i \times e_j \cdot e_k$$

1　　　　−1　　　　0



otherwise

(3) The Kronecker delta (δ_{ij})

$$\delta_{ij} = e_i \cdot e_j \rightarrow \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

(4) 向量外積：

$\underset{\sim}{\boldsymbol{u}} = \langle u_1, u_2, u_3 \rangle$ 與 $\underset{\sim}{\boldsymbol{v}} = \langle v_1, v_2, v_3 \rangle$ 做外積，依據定義為：

$$\underset{\sim}{\boldsymbol{u}} \times \underset{\sim}{\boldsymbol{v}} = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)e_1 - (u_1v_3 - u_3v_1)e_2 + (u_1v_2 - u_2v_1)e_3$$

利用愛因斯坦符號表示即為：

$$\underset{\sim}{\boldsymbol{u}} \times \underset{\sim}{\boldsymbol{v}} = \varepsilon_{ijk}u_jv_k = a_i = \underset{\sim}{\boldsymbol{a}}$$

(5) 梯度（gradient, ∇）：

$$\nabla\phi = \frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} + \frac{\partial\phi}{\partial x_3} = \underset{\sim}{\boldsymbol{u}} \rightarrow \nabla\phi = \frac{\partial\phi}{\partial x_i} = u_i$$

(6) 轉動慣量(moment of inertia)：

$$\begin{cases} I_{11} = \int x_2^2 dA \\ I_{22} = \int x_1^2 dA \\ I_{12} = \int x_1 x_2 dA \\ I_{21} = \int x_2 x_1 dA \end{cases} \rightarrow I_{ij} = \int (\delta_{ij} r^2 - x_i x_j) dA$$

※ 補充(2)與(3)的關係：$\varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$

※ 向量梯度的定義：$\nabla\underset{\sim}{\boldsymbol{u}} = \dfrac{\partial u_i}{\partial x_j}$（二階張量）

二階張量的轉換關係：

若二階張量為 $T_{st} = a_s a_t$，$\overline{T}_{ij} = \overline{a}_i \overline{a}_j$ 且 $\overline{a}_i = L_{is} a_s$，則：

$$\overline{T}_{ij} = \overline{a}_i \overline{a}_j = L_{is} a_s L_{jt} a_t = L_{is} L_{jt} T_{st} \tag{4}$$

因二階張量有兩個自由變數，故一般以矩陣表示之，若以平面座標為例：

$$T_{ij} \rightarrow [T] = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

故（4）式可用矩陣表示成：

$$[\overline{T}] = [L][T][L]^{\mathrm{T}} \tag{5}$$

※這矩陣表示容易出錯，矩陣觀念需特別注意！

練習：

利用愛因斯坦符號推算下列向量運算關係

(1) $\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{c}}$

(2) $(\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}}) \times \underset{\sim}{\boldsymbol{c}}$　　　Hint: $\varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$

(3) $\varepsilon_{ijk} a_j a_k$

(4) $\varepsilon_{ijk}\varepsilon_{ijk}$

Digital Simulation of the Transformation of Plane Stress

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ABSTRACT: In this study, we developed a computer program to simulate the transformation of plane stress by using Visual Basic.NET. We applied the equations of stress transformation to plane stress problems to calculate the stresses with respect to the 1−2 axes, which are rotated counterclockwise through an angle θ about the $x-y$ origin, and showed the visual results on the screen. In addition, we used animation to observe the change of plane stress. This program was then used in teaching courses, such as Mechanics of Materials and Linear Algebra. Use of the software may help students to understand principal stresses, principal axes, Mohr's circle, eigenvalues, eigenvectors, similar matrices, and invariants. ©2008 Wiley Periodicals, Inc. Comput Appl Eng Educ 17: 25–33, 2009; Published online in Wiley InterScience (www.interscience.wiley.com); DOI 10.1002/cae.20180

Keywords: plane stress; principal stresses; principal axes; Mohr's circle; eigenvalues; eigenvectors; similar matrices; invariants

INTRODUCTION

Computer simulations are methods to help students understand a new concept quickly. They have been proven to be an efficient tool both in teaching and self learning [1]. Many researchers have developed computer software for educational purposes. Vidaurre et al. [2] developed simulation programs for curve fitting, wheel motion, and frictions. Lee [3] developed a series of physics simulations. Chimenti and Ochs [4] developed a 3-D simulator for moments of inertia.

This article reports on a digital simulation for the transformation of plane stress. We chose this topic because this simulation can be used to teach many concepts that are not straightforward for students to comprehend.

In linear algebra, students who learn eigenvalue problems for the first time usually have difficulties in understanding the problems. However, eigenvalue problems are very important in science and engineering. So, we designed a simulation to illustrate the eigenvalue problem through plane stress

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transformation. For each different problem, eigenvalues usually have special physical meanings. Moreover, the corresponding eigenvectors usually indicate directions that have special meanings. If a graphical demonstration is given, students will then have a better understanding of the meaning of eigenvalues and eigenvectors.

Teachers generally agree that some concepts in eigenvalue problems often confuse students. Therefore, we try to provide a tool for students to observe the meaning of the problems. With Visual Basic.NET [5] as our platform, we use transformation of plane stress problems to illustrate eigenvalue problems. The stress status in an infinitesimal element of a continuum material can be described as a symmetric matrix [6]. Through graphical illustration, students can easily understand some difficult concepts of eigenvalue problems, such as: similar matrices, physical meaning of eigenvalues, physical meaning of eigenvectors, invariance of trace, invariance of determinant, principal stress, principal direction, maximum shear stress, average stress, etc.

THEORY IN TRANFORMATION OF PLANE STRESS

The theory of transformation of plane stress has been developed for a long time and is well established [6]. We will briefly state it for convenience.

A general three-dimensional state of stress of an infinitesimal element in a continuum material can be expressed by nine stress components σ_{ij} (where i, $j = x$, y, or z), as shown in Figure 1. According to conventional subscript notation, when $i = j$, the stress component is a normal stress; and when $i \neq j$, the stress component is a shear stress. The first subscript represents the direction of the outward normal to the face on which the stress component acts, and the second subscript refers to the direction in which the stress component itself acts [7].

It is to be noted that σ_{ij} is equal to σ_{ji} because of the reciprocity of shear stress. This means that if we write σ_{ij} in matrix form, it will be a symmetric matrix. It is also to be noted that σ_{ii} is a tension stress when its value is positive and σ_{ii} is a compressive stress when its value is negative. On the other hand, σ_{ij} is a shear component no mater whether its value is positive or negative.

In our study, we will concentrate on plane stress problems instead of general three-dimensional problems because two-dimensional plane stress problems are easier to explain to students and easier to illustrate on a computer screen.

In order to describe the stress state of an infinitesimal element of a continuum material under plane stresses, we need a 2 by 2 stress matrix:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \tag{1}$$

The four entries in the stress matrix (1) represent the stresses acting on the element with respect to the $x-y$ coordinates and are shown in Figure 2. The values of σ_{xy} and σ_{yx} are the same, due to reciprocity of shear stresses in Mechanics of Materials. Hence, the stress matrix (1) is symmetric. That is, there are only three independent entries in the stress matrix. The normal stresses acting on the opposite side of the element are equal because the element is vanishingly small.

If we describe the stress state of the element using a new $1-2$ coordinate system, which is rotated

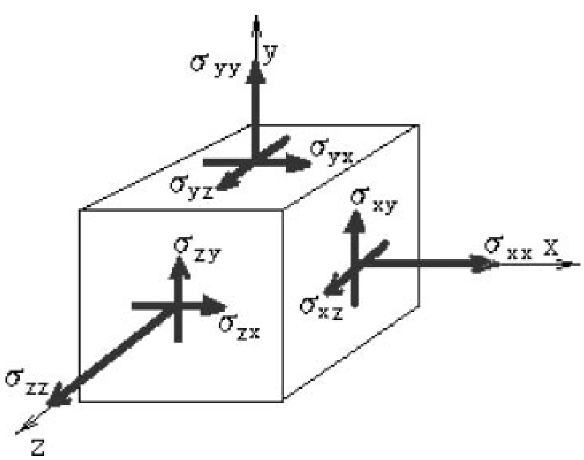

Figure 1 General three-dimensional stresses acting on an infinitesimal element of a continuum material with respect to the $x-y-z$ axes.

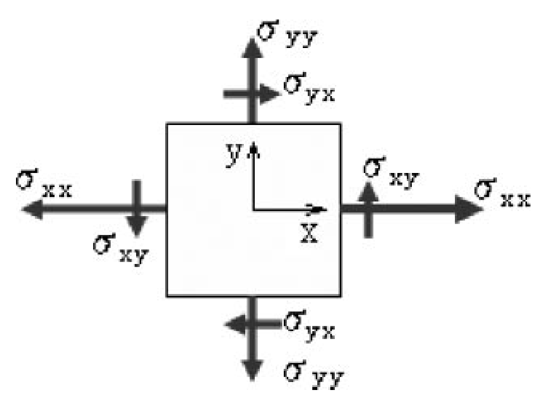

Figure 2 The plane stresses acting on an infinitesimal element of a continuum material with respect to the $x-y$ axes.

counterclockwise through an angle θ about the $x-y$ origin, we need another 2 by 2 symmetric matrix:

$$\sigma' = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \quad (2)$$

The four entries in the stress matrix (2) represent the stresses acting on the element with respect to the 1−2 axes and are shown in Figure 3. Similarly, there are only three independent entries in the stress matrix, since the matrix is symmetric. The 1−2 axes are rotated counterclockwise through an angle θ about the $x-y$ origin.

The two stress matrices mentioned above actually describe the same physical phenomenon with respect to different coordinate systems. Therefore, they are related to each other. We will now derive the relation between them.

Consider the vanishingly small element shown in Figure 4. The inclined surface has an outward normal pointing to the 1 axis, which is rotated from the x axis counterclockwise through an angle θ. From the force equilibrium equations in the horizontal direction and vertical direction, we can obtain the following equations

$$\sigma_{11}\frac{A}{\cos\theta}\cos\theta - \sigma_{12}\frac{A}{\cos\theta}\sin\theta - \sigma_{xx}A - \sigma_{xy}A\frac{\sin\theta}{\cos\theta} = 0 \quad (3)$$

$$\sigma_{11}\frac{A}{\cos\theta}\sin\theta + \sigma_{12}\frac{A}{\cos\theta}\cos\theta - \sigma_{xy}A - \sigma_{yy}A\frac{\sin\theta}{\cos\theta} = 0, \quad (4)$$

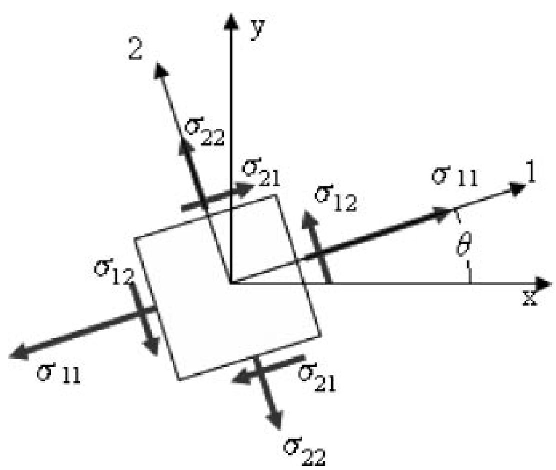

Figure 3 The plane stresses acting on an element with respect to the 1−2 axes. The 1−2 axes are rotated counterclockwise through an angle θ from the original $x-y$ axes.

where A is the area of the vertical surface shown in Figure 4 [8].

Simplifying the above two equations, we have

$$\sigma_{11}\cos\theta - \sigma_{12}\sin\theta = \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta \quad (5)$$

$$\sigma_{11}\sin\theta + \sigma_{12}\cos\theta = \sigma_{xy}\cos\theta + \sigma_{yy}\sin\theta \quad (6)$$

Solving the system of linear Equations (5) and (6), we have

$$\sigma_{11} = \sigma_{xx}\cos^2\theta + 2\sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta \quad (7)$$

$$\sigma_{12} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta - \sigma_{xy}(\sin^2\theta - \cos^2\theta), \quad (8)$$

The value of σ_{22} can be derived from Equation (7) by changing θ to θ + 90°, since the 2 axis is 90° counterclockwise from the 1 axis. Thus, we have

$$\sigma_{22} = \sigma_{xx}\sin^2\theta - 2\sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\cos^2\theta, \quad (9)$$

Equations (7−9) state the relation between the two stress matrices and are widely used in Mechanics of Materials [6−9].

As can be seen from Equation (7), the normal stress is a function of θ. That is, the normal stress varies with angle θ. The maximum and minimum values of the normal stress are called the principal stresses. If we differentiate σ_{11} in Equation (7) with respect to θ and set it equal to zero, we obtain

$$\frac{d\sigma_{11}}{d\theta} = -(\sigma_{xx} - \sigma_{yy})\sin 2\theta + 2\sigma_{xy}\cos 2\theta = 0. \quad (10)$$

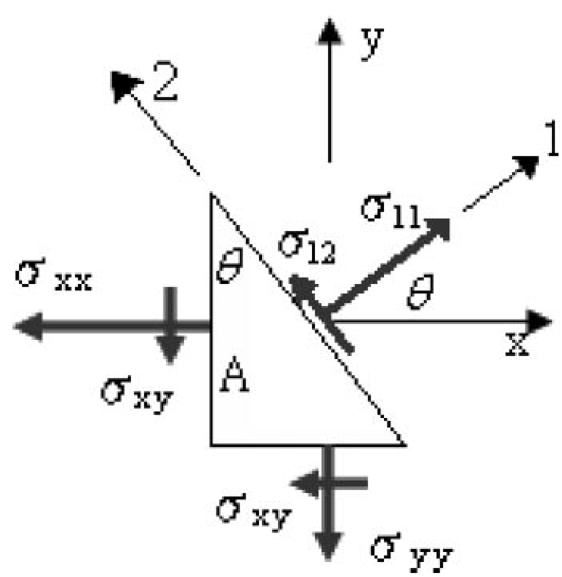

Figure 4 The stresses acting on an element with an inclined side whose normal has an angle +θ from the x axis.

Solving the above equation, we have

$$\tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}, \quad (11)$$

in which θ_p is used in place of θ to denote the angles defining the principle axes.

From Equation (11), two values of θ_p can be obtained. The two θ_p differ with an angle 90° from the definition of tangent. For one of the two values of θ_p, the normal stress is a maximum, and for the other the normal stress is a minimum. That is, when σ_{11} reaches its maximum, σ_{22} will reach a minimum because the two θ_p differ with an angle 90°. Similarly, if σ_{11} is a minimum, σ_{22} will be a maximum. Substituting θ_p into Equation (8), we can also find that when the normal stresses reach maximum or minimum, the shear stress will be zero.

Rearranging Equations (7) and (8), we can obtain

$$\left(\sigma_{11} - \frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \sigma_{12}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2. \quad (12)$$

The above equation is a circle of radius $\sqrt{((\sigma_{xx} - \sigma_{yy})/2)^2 + \sigma_{xy}^2}$ with a center at $((\sigma_{xx} + \sigma_{yy})/2, 0)$. It was developed by Mohr and is called Mohr's circle [6]. Mohr's circle provides an alternative way to solve stress transformation problems. This graphical illustration, although developed more than 100 years ago, is still widely used in engineering. Thus, we also provide this circle in the program for user reference.

Summarizing Equations (7–9) in linear algebraic form, we have the following equation:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad (13)$$

or in simplified form

$$\sigma' = T^{-1}\sigma T$$

Note that the first matrix in the right hand side of Equation (13) is an inverse of a transformation matrix T. In some articles [10], the inverse is replaced by a transpose as follows:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{t} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad (14)$$

Equations (13) and (14) are both correct, because the transformation matrix T in these two equations is an orthogonal matrix. That is, the two columns in the transformation matrix T are orthogonal to each other and have unit length. In linear algebra, the inverse of an orthogonal matrix equals its transpose.

We can also find two invariants I_1 and I_2 from Equations (7–9).

$$I_1 = \sigma_{11} + \sigma_{22} = \sigma_{xx} + \sigma_{yy} = \mathrm{trace}(\sigma) = \mathrm{trace}(\sigma') \quad (15)$$

$$\begin{aligned} I_2 = \sigma_{11}\sigma_{22} - \sigma_{12}^2 &= \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \\ &= \det(\sigma) = \det(\sigma') \end{aligned} \quad (16)$$

From Equation (15), we note that when σ_{11} is a maximum, σ_{22} will be a minimum, since the sum of σ_{11} and σ_{22} is an invariant.

In linear algebra, a general eigenvalue problem can be stated as

$$A\underline{x} = \lambda\underline{x}, \quad (17)$$

where A is an n by n matrix, λ is the eigenvalue of A, and the vector $\underline{x}$ is the corresponding eigenvector of λ. Note that if $\underline{x}$ is a zero vector, Equation (17) is automatically satisfied. However, if $\underline{x}$ is a zero vector, it is not an eigenvector by definition [11].

In our case, the matrix A is replaced by the 2 by 2 stress matrix with respect to the $x-y$ axes. The explicit form of the eigenvalue problem becomes

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (18)$$

Moving the right hand side to the left, we have

$$\begin{bmatrix} \sigma_{xx} - \lambda & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (19)$$

In order to get a nonzero solution of x_1 and x_2 for the above system of linear equations, the determinant of the matrix in Equation (19) must be zero. Thus, we have

$$\lambda^2 - (\sigma_{xx} + \sigma_{yy})\lambda + (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) = 0 \quad (20)$$

Equation (20) is called the characteristic equation. The roots of the equation are the eigenvalues. Because the characteristic equation of the two-dimensional stress matrix is second order, we have two eigenvalues, namely λ_1 and λ_2. Substituting the two eigenvalues into Equation (19), we can solve the corresponding two eigenvectors: $\underline{e_1}$ and $\underline{e_2}$. Thus, we have

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \underline{e_1} = \lambda_1 \underline{e_1} \quad (21)$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \underline{e_2} = \lambda_2 \underline{e_2} \tag{22}$$

Rearranging Equations (21) and (22), we obtain

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \underline{e_1} & \underline{e_2} \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \underline{e_1} & \underline{e_2} \end{bmatrix} \tag{23}$$

Comparing Equations (13) and (23), we note that if we choose an appropriate angle θ, the transformation matrix in Equation (13) will consist of the eigenvectors in its columns, and the left hand side of Equation (13) will be diagonalized.

Since the two stress matrices describe the same phenomenon, they share many common properties. In linear algebra, the two matrices are called similar matrices. They have the same eigenvalues, the same trace, the same determinant, but different eigenvectors. Although their eigenvectors are different in mathematical form, the directions of the eigenvectors are actually the same in real space.

It is also to be noted from linear algebra that the two similar matrices of size 2 by 2 have two invariants, namely: trace and determinant [11]. The two invariants can be derived from Equation (20). Their mathematical expressions are stated in Equations (15) and (16).

For three dimensional stress problems, we will still obtain an equation similar to Equation (13). However, we will have three eigenvalues and three eigenvectors. We will also have three invariants for three dimensional problems.

PROGRAM IMPLEMENTATION

A digital simulation program was then written from the above theory by using Microsoft Visual.Basic.-NET. Figure 5 shows the appearance of the program on the screen. The stress matrix with respect to $x-y$ axes can be entered from the lower left corner. Also, the angle of $1-2$ axes with respect to the original $x-y$ axes can also be entered from the lower left corner. We can now click the ‘‘calculate’’ button to calculate the stress matrix with respect to $1-2$ coordinates. Then the computer will calculate the new stress matrix

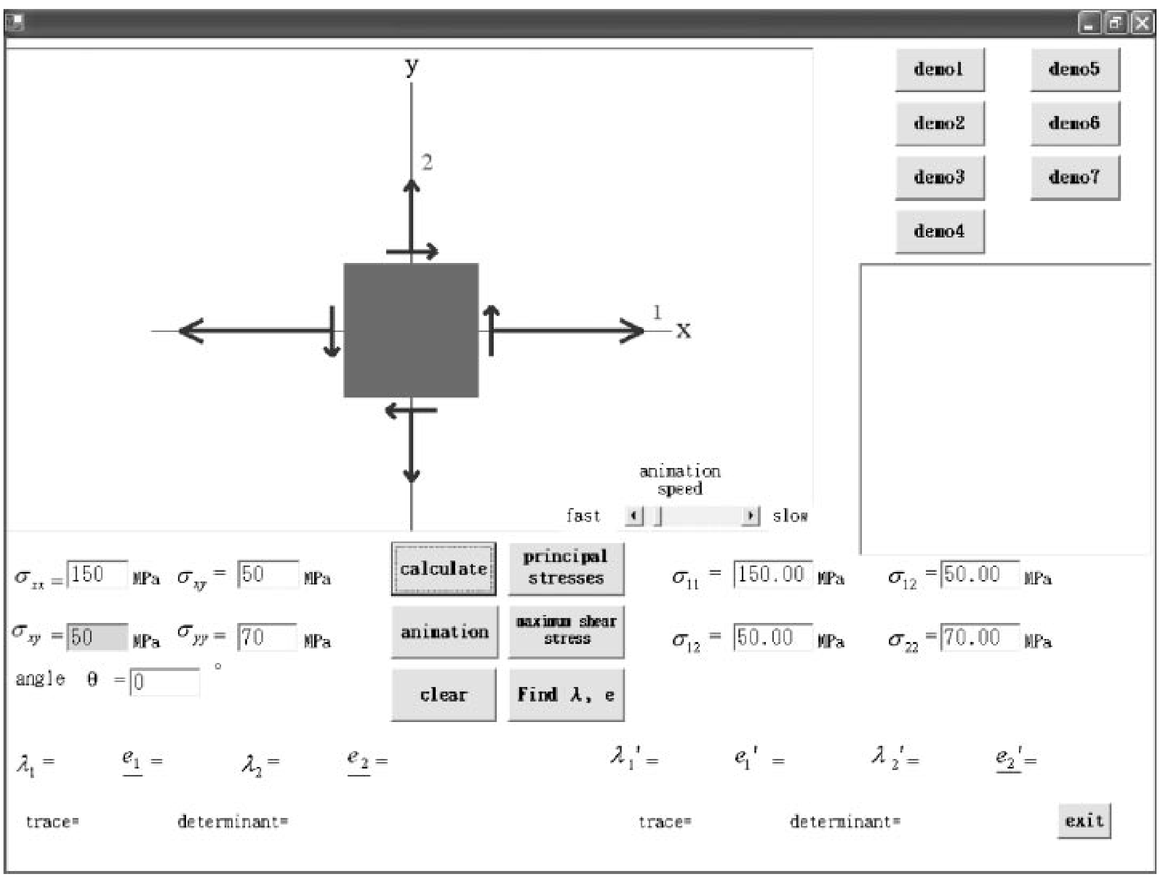

Figure 5 The appearance of the program on the screen. The stress matrix with respect to the $x-y$ axes can be entered from the lower left corner. The angle of $1-2$ axes with respect to the original $x-y$ axes can also be entered from the lower left corner.

automatically. The results of the new stress matrix will be shown on the lower right corner of the screen. A graphical illustration will also be plotted on the large upper left panel.

As can be seen from Figure 5, the original stress matrix and the angle in this particular case is selected to be

$$\sigma = \begin{bmatrix} 150 & 50 \\ 50 & 70 \end{bmatrix} (\text{MPa}), \text{ and } \theta = 0^\circ.$$

If we press the "principal stresses" button, the software will calculate the eigenvalues and the corresponding eigenvectors of the original stress matrix σ. The graphical results and the new stress matrix σ' will be shown in Figure 6. For the stress problem, the eigenvalues represent the principal stresses and the eigenvectors represent the directions of the principal axes.

As can be seen from Figure 6, the shear stress is zero if we describe the stress state of the infinitesimal element with respect to the $1-2$ axes. The stress matrix with respect to $1-2$ axes, shown in the lower right quadrant, is a diagonal matrix. The angle is 25.67° in this particular case. That is, when the $1-2$ axes are rotated 25.67° from the $x-y$ axes counterclockwise, the shear stress will be zero; the normal stresses reach maximum or minimum values and are called the principal stresses.

If we press the "Find λ, e" button, the program will calculate the eigenvalues and the corresponding eigenvectors for both of the matrices and will show the numerical values at the bottom of the screen. As can be seen from Figure 6, the two matrices have the same eigenvalues but different eigenvectors. Although the eigenvectors are different in mathematical form, they point to the same direction in real space. For example, the first eigenvector $\underline{e_1} = (0.90, 0.43)^t$ in the $x-y$ coordinate system points in the same direction as $\underline{e_1'} = (1.00, 0.00)^t$ in the $1-2$ coordinate system. The traces and determinants of the two matrices are also shown at the bottom of the screen.

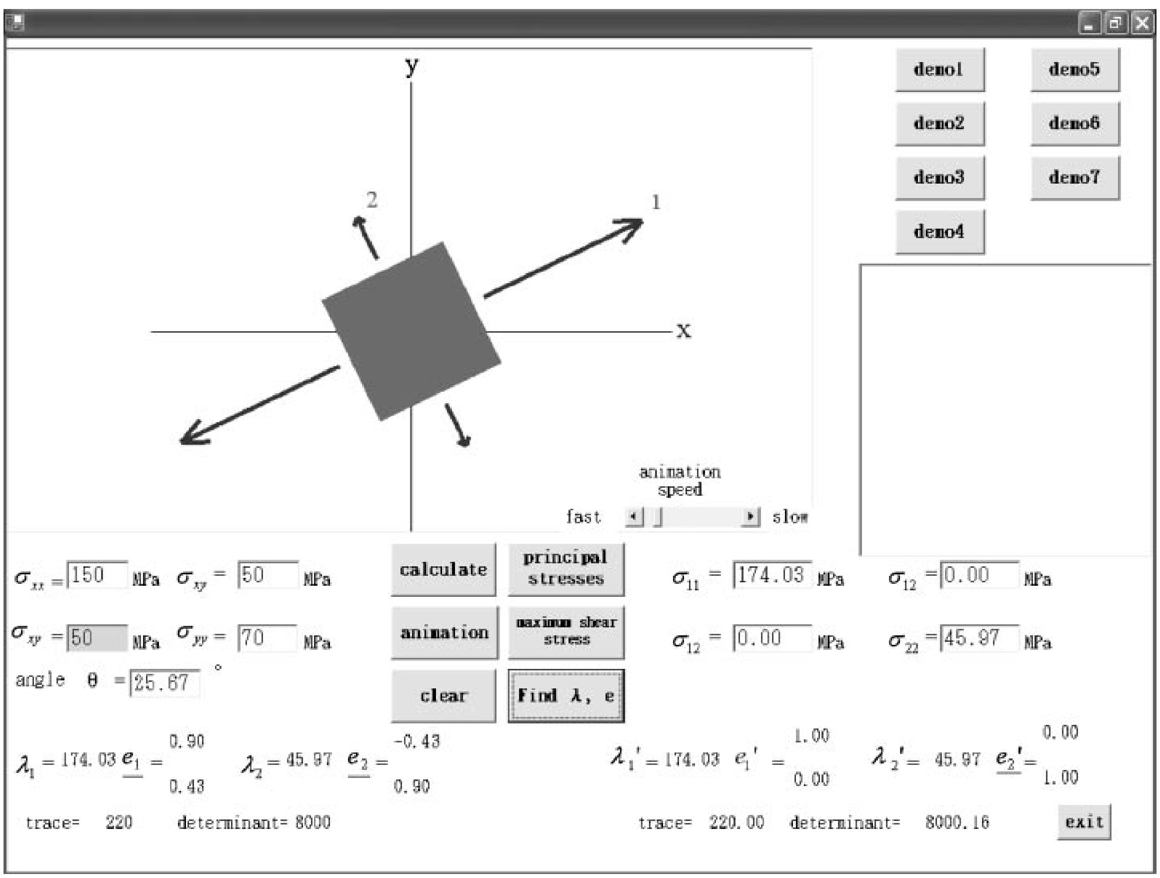

Figure 6 The appearance of program if the "principal stress" and the "Find λ, e" buttons have been pressed. The two stress matrices are similar. The angle is 25.67° in this particular case. The principal stresses are 174.03 and 45.97 MPa.

The value of the angle 25.67° can be calculated from Equation (11):

$$\frac{1}{2}\tan^{-1}\left(\frac{2\times 50}{150-70}\right)\approx 25.67^\circ,$$

or from the eigenvector.

$$\tan^{-1}\left(\frac{0.43}{0.90}\right)\approx 25.67^\circ$$

In this particular case, the principal stresses are 174.03 and 45.97 MPa as shown in Figure 6. The traces and the determinants of the two matrices are the same regardless of the angle θ. Because the trace and determinant do not vary with angle, they are called invariants. The mathematical expressions of the two invariants for this example are:

$$150+70=174.03+45.97$$
$$(150)(70)-(50)(50)\approx(174.03)(45.97)-0^2$$

If we press the ‘‘maximum shear stress’’ button, the software will calculate the direction of the axes and the corresponding stresses. The graphical results are also shown in Figure 7. As can been seen from Figure 7, the rotation angle is −19.33°. The shear stress reaches a maximum value of 64.03 MPa. The normal stresses with respect to the 1−2 axes are equal and are both equal to 110 MPa. This value of normal stresses is the average of the two original normal stresses. Thus, when the shear stress reaches its maximum, the normal stresses are equal to the average stress. The mathematical expressions of the two invariants for this example are:

$$150+70=110+110$$
$$(150)(70)-(50)(50)\approx(110)(110)-(64.03)(64.03)$$

If we press the ‘‘animation’’ button, the software will rotate the 1−2 axes automatically. The stresses for every rotational angle will be shown in the lower right quadrant and the graphical presentation will also

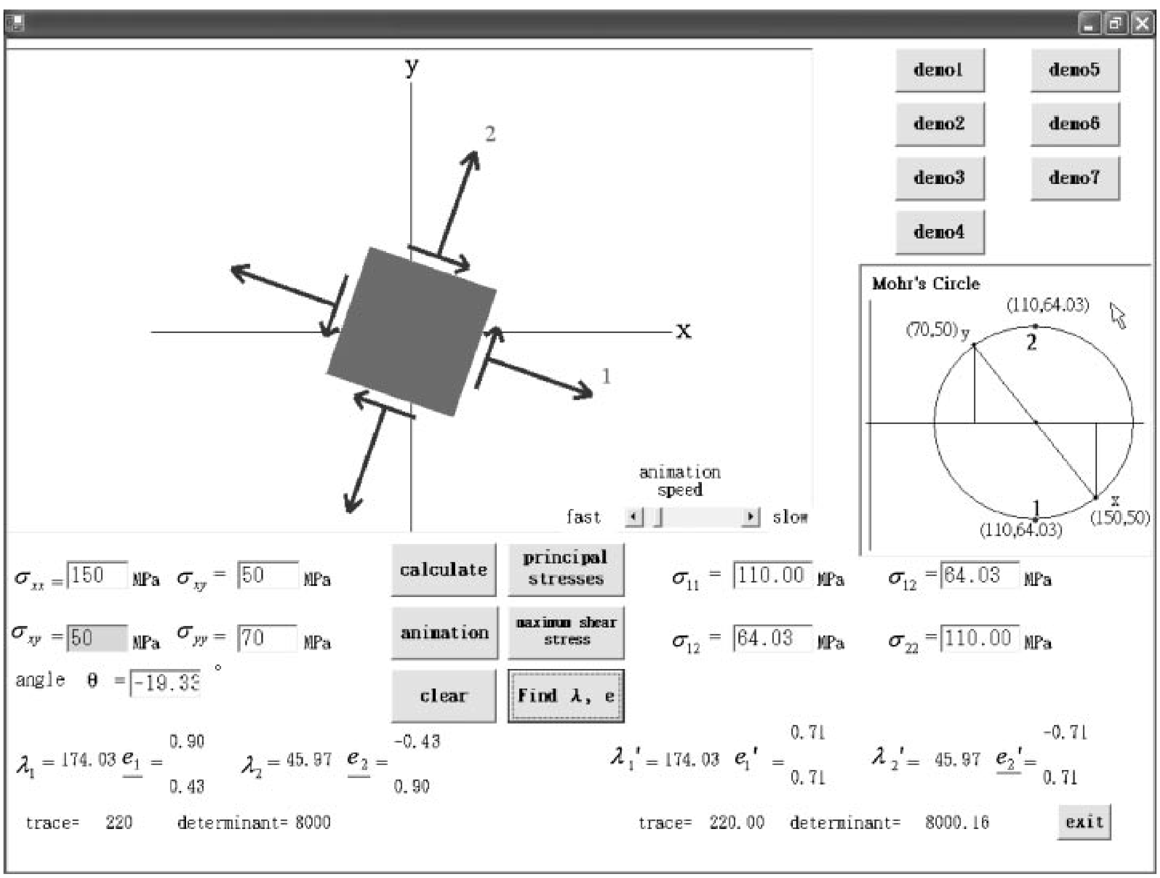

Figure 7 The appearance of the program if the ‘‘maximum shear stress’’ button is pressed. The shear stress reaches a maximum value of 64.03 MPa, while the two normal stresses are equal to 110 MPa. The rotational angle is −19.33°. The two stress matrices are similar.

be plotted. The rotational speed can be controlled by the "animation speed" controller.

Because some students are not familiar with the topics, they may have difficulties when entering the initial values. Seven "demo" buttons are provided in the upper right corner. If the students press one of the seven buttons, the software will enter values of a special case automatically. The seven demo buttons give case of pure tension, pure compression, pure shear, and some general cases.

The Mohr's circle will also be provided in the lower right corner for every special case if the mouse moves over the lower right Mohr's circle plotting panel. Mohr's circle is an alternative way to solve plane stress transformation problems. In Mohr's circle, the horizontal axis represents normal stress while the vertical axis represents shear stress. A shear stress, which tends to rotate the infinitesimal element clockwise, will be drawn above the horizontal axis. As can be seen from Figure 7, a point denoted as "x" has coordinates (150,50) on the Mohr's circle. This means that the normal stress is 150 MPa, and the shear stress is 50 MPa, which tends to rotate the infinitesimal element counterclockwise. Therefore, the point "x" is drawn below the horizontal axis. If we rotate 38.66° (twice of 19.33°) clockwise from the "x" point, we will reach the bottom of the circle, denoted as "1" in the Mohr's circle panel, which gives us the normal stress as 110 MPa and shear stress as 64.03 MPa counterclockwise.

STUDENT EXPERIENCE

This software was used to teach courses such as Mechanics of Materials and Linear Algebra. The students used different sets of stress matrix values and rotational angle values to observe the relations among quantities. The preliminary results are quite satisfactory. Most students have very positive responses to this software.

In Mechanics of Materials, students seem to have higher interest levels since a graphical presentation is provided. Last year, when we did not use this software in teaching Mechanics of Materials, the average feedback from the students at the end of the semester was 3.94 on a 5 point scale. After we used this software this year, the feedback was 4.12. Students seem to have a better understanding of plane stresses, principal stresses, principal axes, maximum shear stresses, and transformation of stresses. Then we include the stress transformation problems in the final exam. However, the improvement in the final exam is small. Last year, the average of the final exam was 63.5. This year the average is 64.1. This is probably because we teach the two approaches in solving stress transformation problems. Last year's students concentrated on traditional Equations (7–9). Meanwhile, this year's students study both the traditional approach and linear algebra approach (13), so the improvement is small. Although the difference in exams is minor, the linear algebra approach, Equation (13), is much easier to memorize than Equations (7–9). Probably, this software may help students more in the long term.

In Linear Algebra, students will realize that eigenvalue problems can be applied in stress transformation. Students will have a higher motivation to learn how to solve eigenvalue problems. Before we used this software in teaching Linear Algebra, the feedback from the students at the end of the semester was 3.84 on a 5 point scale. After we used this software, the feedback was 4.22. The students tend to have a better understanding of eigenvalues, eigenvectors, symmetric matrices, similar matrices, orthogonality of eigenvectors, linear transformation, invariants, and diagonalization. Moreover, the improvement in the final exam is quite significant. In our course schedule, eigenvalue problems do not appear in the midterm exam, but they play a major part in the final exam. Last year, the average of the final exam was 65.5. This year, the average was 72.3. This is probably because students think eigenvalue problems are important.

In Taiwan, Students usually take Linear Algebra during their first year and take Mechanics of Materials during their second or third year in university. It is to be noted that students in Taiwan take Linear Algebra from the faculty of their own department, not from the faculty of mathematics. Therefore, students who are taking Linear Algebra in our departments understand that they will have to learn plane stress transformation at a later time.

This program is designed for students in engineering who are studying Mechanics of Materials or Linear Algebra for the first time in the university level. If this program is used to teach students in other fields, such as mathematics, the instructors may have to explain concepts in mechanics very clearly.

It is generally not easy to compare the responses from students because the students are different from year to year. The analysis of responses from students is still quite preliminary. We intend to continue to study students' responses to improve our software.

CONCLUSIONS

This computer program provides a powerful tool for students to understand transformation to plane stress problems. It calculates the stresses with respect to the

1−2 axes, which are rotated counterclockwise through an angle θ about the $x-y$ origin, and shows the visual results on the screen. In addition, students can use animation to observe the change of plane stress. This program can also find the eigenvalues and eigenvectors of the two stress matrices. The program was used in teaching courses, such as Mechanics of Materials and Linear Algebra. The students' motivation was highly increased. The students tended to have higher interest levels and tended to have a better understanding of principal stresses, principal directions, Mohr's circle, eigenvalues, eigenvectors, similar matrices, and invariants.

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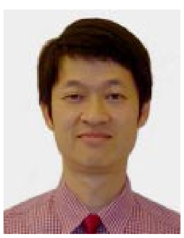
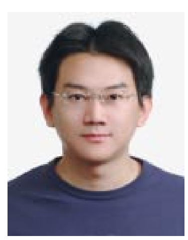
BIOGRAPHIES



Wei-Pin Lee received a BS degree in mechanical engineering from National Taiwan University in 1986, an MS degree in mechanical engineering from the University of Michigan in 1990, and a PhD degree in mechanical engineering from the University of Michigan in 1992. Since 1992 he has been teaching mathematics and applied physics in the Department of Fiber and Composite Materials, Feng Chia University, Taiwan. He is the author of a series of teaching software, "WEE Play Physics."

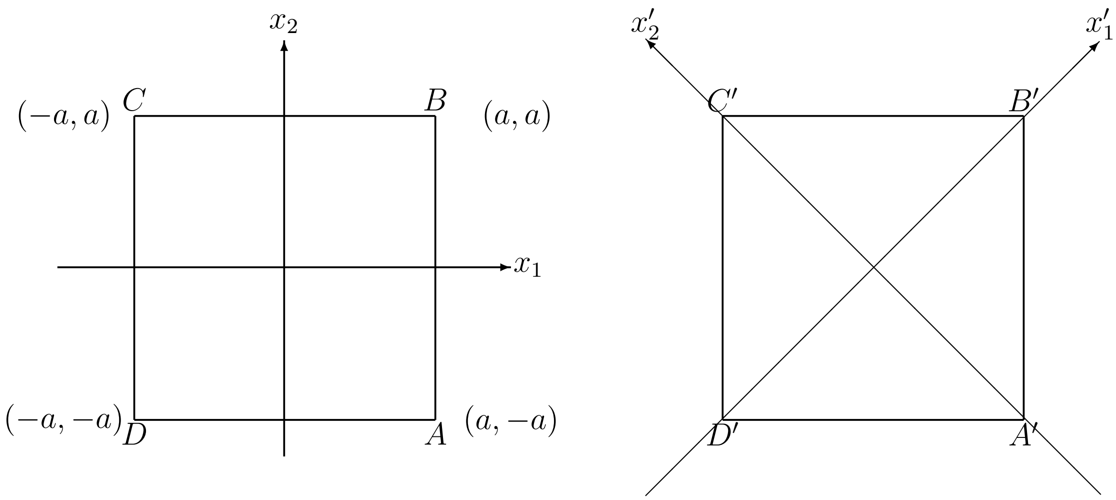
Chang-Hsuan Chiu received a BS degree in textile engineering in 1981, an MS degree in 1983, and a PhD degree in 1987 from Feng Chia University. He worked at Aerospace Industrial Development Corp., Taiwan, from 1987 to 1993. Since 1993, he has been teaching in the Department of Fiber and Composite Materials, Feng Chia University, Taiwan. He is currently the vice dean of the College of Engineering.

Chung-Li Hwan received a BS degree in mechanical engineering from National Tsing Hua University, Taiwan in 1978, an MS degree in mechanical engineering from the Arizona State University in 1986, and a PhD degree in mechanical engineering from the University of Michigan in 1992. Since 1992 he has been teaching engineering mathematics and the finite element methods in the Department of Mechanical and Computer Aided Engineering, Feng Chia University, Taiwan.



John Ma received a BS degree in textile engineering in 2003 and an MS degree in 2006 from Feng Chia University. His main research interests are mechanics of composite materials and computer simulations.

Scalar, Vector and Tensor



x_1-x_2 coordinate system x'_1-x'_2 coordinate system

1. Square $ABCD$ can be described in x_1x_2 and $x'_1x'_2$ coordinates.

   (1). Express the coordinates for A', B', C' and D' in terms of x'_1, x'_2 coordinate system.

   (2). Determine the length of $\overline{AC}$ and $\overline{BD}$ in both coordinates. Discuss the result.

   (3). Determine the vectors of $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in both coordinates. Also, find the inner and outer products of $\overrightarrow{AC}$ and $\overrightarrow{BD}$. Any difference between the two coordinate discriptions? Find the area of $ABCD$.

   (4). If a vector can be expressed by (v_1, v_2) and (v'_1, v'_2) in $x_1 - x_2$ and $x'_1 - x'_2$ coordinate system, respectively, find the matrix T if

   $$\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

   (5). Find I_{11}, I_{12}, I_{21} and I_{22} if $I_{ij} = \int_A x_i x_j dA$.

   (6). Find $I_{1'1'}$, $I_{1'2'}$, $I_{2'1'}$ and $I_{2'2'}$ if $I_{i'j'} = \int_A x_{i'} x_{j'} dA$.

   (7). If moment of inertia can be expressed by $I_{11}, I_{12}, I_{21}, I_{22}$ and $I_{1'1'}, I_{1'2'}, I_{2'1'}, I_{2'2'}$ in $x_1 - x_2$ and $x'_1 - x'_2$ coordinate system, respectively, find the matrix A if

   $$\begin{bmatrix} I_{1'1'} & I_{1'2'} \\ I_{2'1'} & I_{2'2'} \end{bmatrix} = A^T \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} A$$

   (8). Compare $I_{11} + I_{22}$ with $I_{1'1'} + I_{2'2'}$

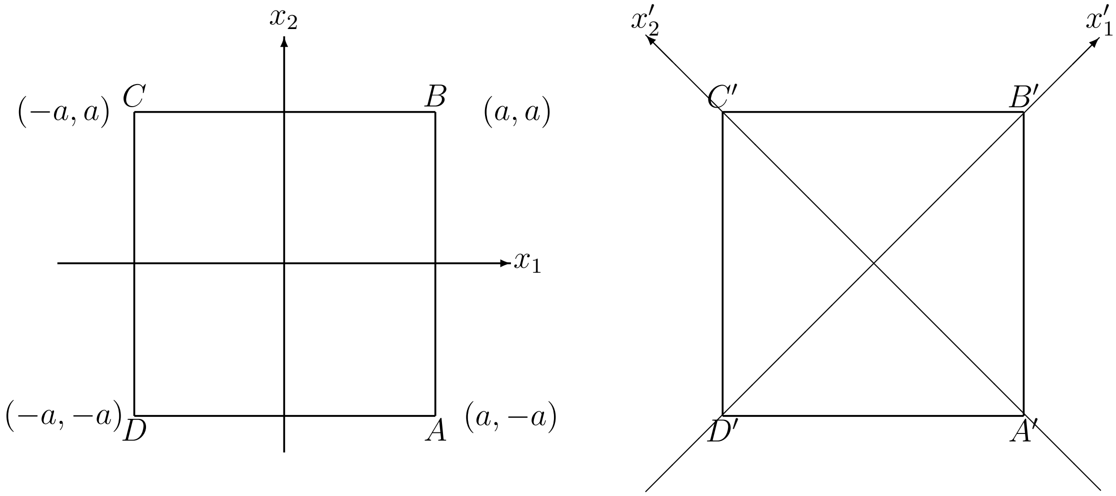
   (9). Compare

   $$det \begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix} \quad with \quad det \begin{vmatrix} I_{1'1'} & I_{1'2'} \\ I_{2'1'} & I_{2'2'} \end{vmatrix}$$

   (10). Find the eigenvalues and eigenvectors for the two matrix.

2. Classify all the calulated quantities to scalar, vector or rank-2 tensor.

Tensor rank	name	physical quantity
0	scalar	
1	vector	
2	tensor(2)	



x_1-x_2 coordinate system x'_1-x'_2 coordinate system

(1). Scalar transformation

$$c = c$$

(2). If a vector can be expressed by (v_1, v_2) and (v'_1, v'_2) in $x_1 - x_2$ and $x'_1 - x'_2$ coordinate system, respectively, find the matrix T if

$$\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = [T] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where

$$[T] = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$$

(3). If moment of inertia can be expressed by $I_{11}, I_{12}, I_{21}, I_{22}$ and $I_{1'1'}, I_{1'2'}, I_{2'1'}, I_{2'2'}$ in $x_1 - x_2$ and $x'_1 - x'_2$ coordinate system, respectively, find the matrix A if

$$\begin{bmatrix} I_{1'1'} & I_{1'2'} \\ I_{2'1'} & I_{2'2'} \end{bmatrix} = [T]^t \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} [T]$$

Are they tensors ?

(a) $(x_2\ ,-x_1)$ (b) $(x_2\ ,x_1)$ (c) (x_1^2, x_2^2)

(d) $[T_{ij}] = \begin{bmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{bmatrix}$

(e) Stress, strain, traction, displacement vector, force vector,

(f) moment of inertia ?

(g) $I_{ij} = \int (r^2\delta_{ij} - x_i x_j)dA$ dynamics

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1). Find eigenvalues of A.
(2). Find eigenvectors of A.
(3). Find $A^3 - A^2 - A + I$.
(4). Find C and D, such that $AC = CD$.
(5). Find e^A.
(6). Find A^{100}.
(7). Find $sin(A)$.
(8). Find $A^{1/2}$.
(9). Find rank of A.
(10). Find nullity of A.
(11). Is A singular ?

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

(1). Find eigenvalues of A.
(2). Find eigenvectors of A.
(3). Find C and D, such that $AC = CD$.

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

(1). Find eigenvalues of A.
(2). Find eigenvectors of A.
(3). Find C and D, such that $AC = CD$.
(4). Find A^{100}.

Why matrice ?

1. Force equilibrium
2. Flow equilibrium
3. Least square
4. Quadratic form
5. Grapph theory
6. Game theory
7. Geometric interpretation
8. Numerical method
9. Magic squares

$$\begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

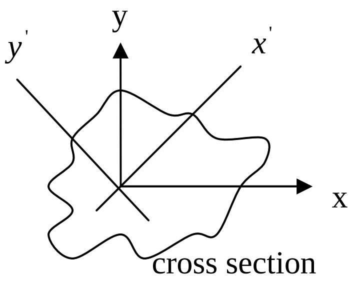
10. Linear transformation ($\mathbf{y} = A\mathbf{x}$)

Matrice operation:

1. Addition (C=A+B)
2. Multilpication (E=AB)
3. Decomposition (F=RU=VR)
4. Transpose ($G = C^T$)
5. Eigenvalues and eigenvectors ($A\mathbf{x} = \lambda\mathbf{x}$)
6. Similar matrices and diagonalization (AC=CD)
7. Inverse, determinant and Cramer's rule ($A^{-1}, detA$)
8. Symmetric matrices and orthogonal diagonalization ($A^T = A = \Phi^T D \Phi$)
9. Singularity, rank and nullity
10. e^A, matrice differential equation and Cayley-Hamilton theorem ($\dot{\mathbf{x}} = A\mathbf{x}$)

Eigenvalues & Eigenvectors

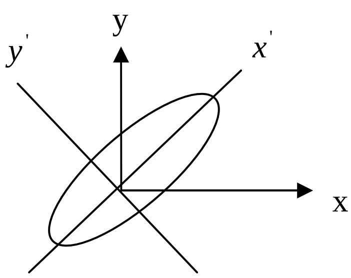
(1) moment of inertia (mechanics of materials)



major axis

minor axis

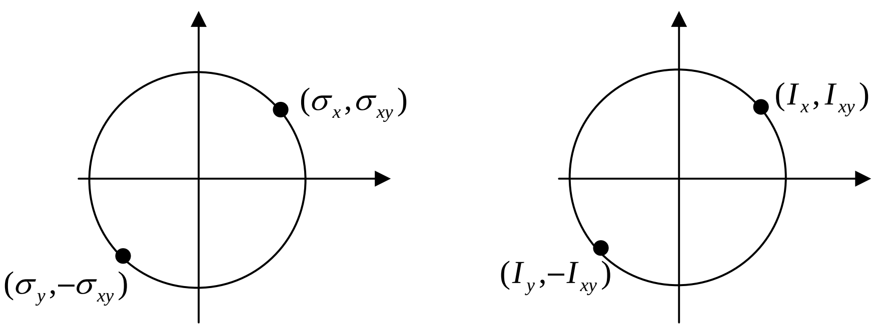
(2) coordinate transformation



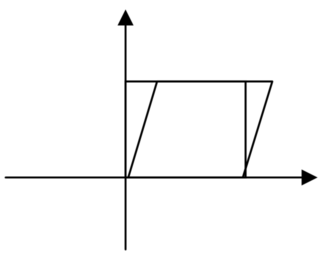
$$ax^2 + bxy + cy^2 = d^2$$

$$\Rightarrow \frac{x^2}{\bar{a}^2} + \frac{y^2}{\bar{b}^2} = 1$$

(3) Stress & strain(Mohr circle)



(4) Deformation



(a) Rotating and stretching

(b) Stretching and rotating

(5) Free vibration (natural frequency and natural mode)　Two DOFs

frequency corresponds to a solution where the string and rod are moving with opposite phase and $x_1 : x_2 = 9.359 : -16.718$. The two situations are shown in figure 9.1.

In connection with quadratic forms it was shown in section 8.17 how to make a change of coordinates such that the matrix for a particular form becomes diagonal. In exercise 9.6 a method is developed for diagonalising simultaneously two quadratic forms (though the transformation matrix may not be orthogonal). If this process is carried out for A and B in a general system undergoing stable oscillations, the kinetic and potential energies in the new variables η_i take the forms

$$T = \sum_i \mu_i \dot{\eta}_i^2 = \dot{\eta}^{\mathrm{T}} \mathsf{M} \dot{\eta}, \qquad \mathsf{M} = \mathrm{diag}\,(\mu_1, \mu_2, \ldots, \mu_N), \tag{9.11}$$

$$V = \sum_i \nu_i \eta_i^2 = \eta^{\mathrm{T}} \mathsf{N} \eta, \qquad \mathsf{N} = \mathrm{diag}\,(\nu_1, \nu_2 \ldots, \nu_N), \tag{9.12}$$

and the equations of motion are the *uncoupled* equations

$$\mu_i \ddot{\eta}_i + \nu_i \eta_i = 0, \qquad i = 1, 2, \ldots, N. \tag{9.13}$$

Clearly a simple renormalisation of the η_i can be made that reduces all the μ_i in (9.11) to unity. When this is done the variables so formed are called *normal coordinates* and equations (9.13) the *normal equations*.

When a system is executing one of these simple harmonic motions it is said to be in a *normal mode*, and once started in such a mode it will repeat its motion exactly after each interval of $2\pi/\omega_i$. Any arbitrary motion of the system may be written as a superposition of the normal modes, and each component mode will execute harmonic motion with the corresponding eigenfrequency; however, unless by chance the eigenfrequencies are in integer relationship, the system will never return to its initial configuration after any finite time interval.

As a second example we will consider a number of masses coupled together by springs. For this type of situation the potential and kinetic energies are automatically quadratic functions of the coordinates and their derivatives, provided the elastic limits of the springs are not exceeded, and the oscillations do not have to be vanishingly small for the analysis to be valid.

►*Find the normal frequencies and modes of oscillation of three particles of masses m, μm, m connected in that order in a straight line by two equal light springs of force constant k. (This arrangement could serve as a model for some linear molecules, e.g.* CO_2.*)*

The situation is shown in figure 9.2; the coordinates of the particles, x_1, x_2, x_3, are measured from their equilibrium positions, at which the springs are neither extended nor compressed.

The kinetic energy of the system is simply

$$T = \tfrac{1}{2} m \left(\dot{x}_1^2 + \mu \dot{x}_2^2 + \dot{x}_3^2 \right),$$

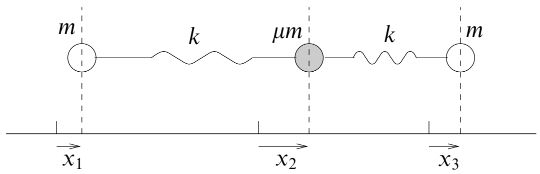


Figure 9.2 Three masses m, μm and m connected by two equal light springs of force constant k.

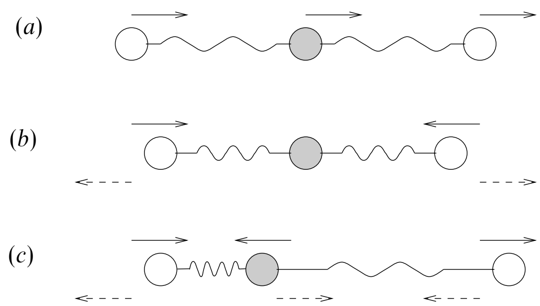


Figure 9.3 The normal modes of the masses and springs of a linear molecule such as CO_2. (*a*) $\omega^2 = 0$; (*b*) $\omega^2 = k/m$; (*c*) $\omega^2 = [(\mu+2)/\mu](k/m)$.

whilst the potential energy stored in the springs is

$$V = \tfrac{1}{2}k\left[(x_2 - x_1)^2 + (x_3 - x_2)^2\right].$$

The kinetic- and potential-energy symmetric matrices are thus

$$\mathsf{A} = \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{B} = \frac{k}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

From (9.10), to find the normal frequencies we have to solve $|\mathsf{B} - \omega^2\mathsf{A}| = 0$. Thus, writing $m\omega^2/k = \lambda$, we have

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\mu\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0,$$

which leads to $\lambda = 0$, 1 or $1 + 2/\mu$. The corresponding eigenvectors are respectively

$$\mathsf{x}^1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathsf{x}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathsf{x}^3 = \frac{1}{\sqrt{2 + (4/\mu^2)}}\begin{pmatrix} 1 \\ -2/\mu \\ 1 \end{pmatrix}.$$

The physical motions associated with these normal modes are illustrated in figure 9.3. The first, with $\lambda = \omega = 0$ and all the x_i equal, merely describes bodily translation of the whole system, with no (i.e. zero-frequency) internal oscillations.

In the second solution the central particle remains stationary, $x_2 = 0$, whilst the other two oscillate with equal amplitudes in antiphase with each other. This motion, which has frequency $\omega = (k/m)^{1/2}$, is illustrated in figure 9.3(*b*).

The final and most complicated of the three modes has frequency $\omega = \{[(\mu +$

$2)/\mu](k/m)\}^{1/2}$, and involves a motion of the central particle which is in antiphase with that of the two outer ones and which has an amplitude $2/\mu$ times as great. In this motion (see figure 9.3(c)) the two springs are compressed and extended in turn. We also note that in the second and third normal modes the centre of mass of the molecule remains stationary. ◀

9.2 Symmetry and normal modes

It will have been noticed that the system in the above example has an obvious symmetry under the interchange of coordinates 1 and 3: the matrices A and B, the equations of motion and the normal modes illustrated in figure 9.3 are all unaltered by the interchange of x_1 and $-x_3$. This reflects the more general result that for each physical symmetry possessed by a system, there is at least one normal mode with the same symmetry.

The general question of the relationship between the symmetries possessed by a physical system and those of its normal modes will be taken up more formally in chapter 25 where the representation theory of groups is considered. However, we can show here how an appreciation of a system's symmetry properties will sometimes allow its normal modes to be guessed (and then verified), something that is particularly helpful if the number of coordinates involved is greater than two and the corresponding eigenvalue equation (9.10) is a cubic or higher-degree polynomial equation.

Consider the problem of determining the normal modes of a system consisting of four equal masses M at the corners of a square of side $2L$, each pair of masses being connected by a light spring of modulus k that is unstretched in the equilibrium situation. As shown in figure 9.4, we introduce Cartesian coordinates x_n, y_n, with $n = 1, 2, 3, 4$, for the positions of the masses and denote their displacements from their equilibrium positions $\mathbf{R}_n$ by $\mathbf{q}_n = x_n\mathbf{i} + y_n\mathbf{j}$. Thus

$$\mathbf{r}_n = \mathbf{R}_n + \mathbf{q}_n \quad \text{with} \quad \mathbf{R}_n = \pm L\mathbf{i} \pm L\mathbf{j}.$$

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The potential energy matrix B is much more difficult to calculate and involves, for each pair of values m, n, evaluating the quadratic approximation to the expression

$$b_{mn} = \tfrac{1}{2}k\left(|\mathbf{r}_m - \mathbf{r}_n| - |\mathbf{R}_m - \mathbf{R}_n|\right)^2.$$

Expressing each $\mathbf{r}_i$ in terms of $\mathbf{q}_i$ and $\mathbf{R}_i$ and remembering that $|\mathbf{R}_m - \mathbf{R}_n| \gg$

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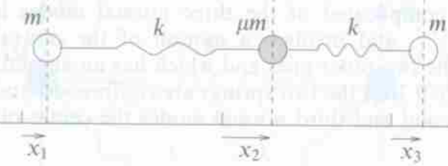

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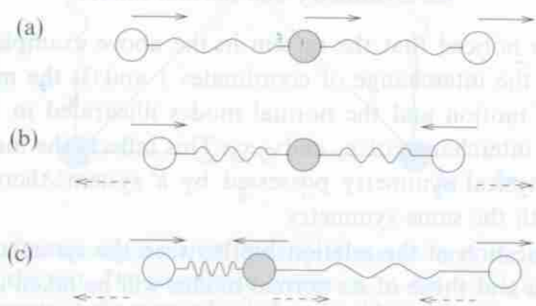

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Expressing each $\mathbf{r}_i$ in terms of $\mathbf{q}_i$ and $\mathbf{R}_i$ and making the normal assumption that

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Fredholm alternative theorem：

Solve $A\underset{\sim}{x}=\underset{\sim}{b}$，

(1) $\det A \neq 0$，$\underset{\sim}{x}$ can be obtained by Cramer's rule.

(2) $\det A = 0$，

$\underset{\sim}{b}\cdot\underset{\sim}{\psi}=0 \rightarrow$ infinite solution

$\underset{\sim}{b}\cdot\underset{\sim}{\psi}\neq 0 \rightarrow$ no solution

where $A^T\underset{\sim}{\psi}=\underset{\sim}{0}$.

Solve $\begin{bmatrix} 12 & 6 & -12 & 6 \\ -6 & -4 & 6 & -2 \\ 12 & 6 & -12 & 6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}=\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix},\begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix},\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix},\begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix},\begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}$ by using the Fredholm alternative theorem.

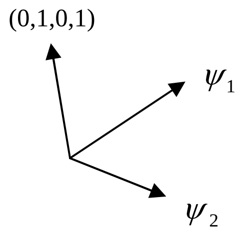
Let $A=\begin{bmatrix} 12 & 6 & -12 & 6 \\ -6 & -4 & 6 & -2 \\ 12 & 6 & -12 & 6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$，$\det(A)=0$

$$A^T\underset{\sim}{\psi}=\underset{\sim}{0} \rightarrow \begin{bmatrix} 12 & -6 & 12 & 6 \\ 6 & -4 & 6 & 2 \\ -12 & 6 & -12 & -6 \\ 6 & -2 & 6 & 4 \end{bmatrix}\underset{\sim}{\psi}=\underset{\sim}{0}$$

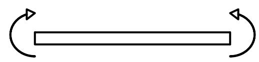
$$\rightarrow \underset{\sim}{\psi}=C_1\begin{Bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{Bmatrix}+C_2\begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}=C_1\underset{\sim}{\psi}_1+C_2\underset{\sim}{\psi}_2 \text{，} C_1,C_2\in\Re$$

(1) $\underset{\sim}{\psi}_1 \cdot \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \neq 0$ ， $\underset{\sim}{\psi}_2 \cdot \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = 0 \ \rightarrow$ no solution

(2) $\underset{\sim}{\psi}_1 \cdot \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \neq 0$ ， $\underset{\sim}{\psi}_2 \cdot \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = 0 \ \rightarrow$ no solution



(3) $\underset{\sim}{\psi}_1 \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} = 0$ ， $\underset{\sim}{\psi}_2 \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} = 0 \ \rightarrow$ infinite solution



$$A\underset{\sim}{x} = \underset{\sim}{b} \ \rightarrow \ \underset{\sim}{x} = C_3 \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} + C_4 \begin{Bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2} \\ -1 \\ 0 \\ 0 \end{Bmatrix} \text{，} C_3, C_4 \in \Re$$

(4) $\underset{\sim}{\psi}_1 \cdot \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} \neq 0$ ， $\underset{\sim}{\psi}_2 \cdot \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} \neq 0 \ \rightarrow$ no solution

(5) $\underset{\sim}{\psi}_1 \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} \neq 0$ ， $\underset{\sim}{\psi}_2 \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} = 0 \ \rightarrow$ no solution

◎ Cayley-Hamilton 定理

$$A\vec{x} = \lambda\vec{x} \Leftrightarrow (A - \lambda I)\vec{x} = 0$$

$$\det|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

$$\lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - c_3\lambda^{n-2} \ldots (-1)^n c_n = 0$$

$$\begin{cases} \lambda_1{}^n - c_1\lambda_1{}^{n-1} + c_2\lambda_1{}^{n-2} - c_3\lambda_1{}^{n-2} \ldots (-1)^n c_n = 0 \\ \lambda_2{}^n - c_1\lambda_2{}^{n-1} + c_2\lambda_2{}^{n-2} - c_3\lambda_2{}^{n-2} \ldots (-1)^n c_n = 0 \\ \vdots \\ \lambda_n{}^n - c_1\lambda_n{}^{n-1} + c_2\lambda_n{}^{n-2} - c_3\lambda_n{}^{n-2} \ldots (-1)^n c_n = 0 \end{cases}$$

將上式寫成矩陣型式

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}^n - c_1 \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}^{n-1} + c_2 \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}^{n-2} - c_3 \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}^{n-2} \ldots (-1)^n c_n \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = [0]_{n\times n}$$

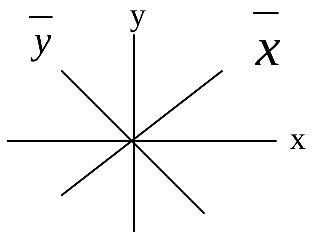
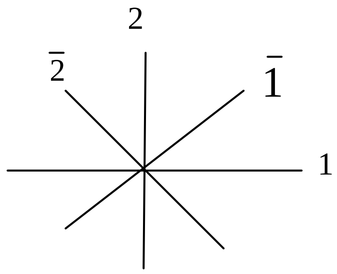
$$D^n - c_1 D^{n-1} + c_2 D^{n-2} - c_3 D^{n-2} \ldots (-1)^n c_n I = [0]_{n\times n}$$

左邊同乘 P ，右邊同乘 P^{-1}

$$PD^n P^{-1} - c_1 PD^{n-1}P^{-1} + c_2 PD^{n-2}P^{-1} - c_3 PD^{n-2}P^{-1} \ldots (-1)^n c_n PIP^{-1} = [0]_{n\times n}$$

$$A^n - c_1 A^{n-1} + c_2 A^{n-2} - c_3 A^{n-2} \ldots (-1)^n c_n I = [0]_{n\times n}$$

Moment of inertia<**數學版與靜、動、材力之比較**>

<table>
<tr><th>Engineering moment of inertia</th><th>Mathematical moment of inertia</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td><Definition>
$I_{xx} = \int_A y^2 dA$
$I_{xy} = \int_A xydA$
$I_{yx} = \int_A yxdA$
$I_{yy} = \int_A x^2 dA$</td><td><Definition>
$I_{ij} = \iint_A \left(r^2 d_{ij} - x_i x_j\right) dA$
$I_{11} = \int_A {x_2}^2 dA$
$I_{12} = -\int_A x_1 x_2 dA$
$I_{21} = -\int_A x_2 x_1 dA$
$I_{22} = \int_A {x_1}^2 dA$</td></tr>
<tr><td><Relation>
$I_{\overline{xx}} = I_{xx}\cos^2 q + I_{yy}\sin^2 q$
$-2I_{xy}\sin q\cos q$

$I_{\overline{yy}} = I_{xx}\sin^2 q + I_{yy}\cos^2 q$
$+2I_{xy}\sin q\cos q$

$I_{\overline{xy}} = I_{xx}\sin q\cos q - I_{yy}\sin q\cos q$
$+I_{xy}\left(\cos^2 q - \sin^2 q\right)$</td><td><Relation>
$\bar{I}_{11} = I_{11}\cos^2 q + I_{22}\sin^2 q$
$+2I_{12}\sin q\cos q$

$\bar{I}_{22} = I_{11}\sin^2 q + I_{22}\cos^2 q$
$-2I_{12}\sin q\cos q$

$\bar{I}_{12} = -I_{11}\sin q\cos q + I_{22}\sin q\cos q$
$+I_{12}\left(\cos^2 q - \sin^2 q\right)$</td></tr>
<tr><td>?</td><td>$\begin{bmatrix}\bar{I}_{11} & \bar{I}_{12}\\ \bar{I}_{21} & \bar{I}_{22}\end{bmatrix} =$
$\begin{bmatrix}\cos q & \sin q\\ -\sin q & \cos q\end{bmatrix}\begin{bmatrix}I_{11} & I_{12}\\ I_{21} & I_{22}\end{bmatrix}\begin{bmatrix}\cos q & \sin q\\ -\sin q & \cos q\end{bmatrix}^T$</td></tr>
<tr><td>**參考文獻**
Gere 材力課本</td><td>**海大河工系 工數 陳正宗講義**</td></tr>
</table>

Feb.27/2004 **註：(工程剪應變與剪應變張量亦有差異)**

檔名：inertia.doc　Henry 製表

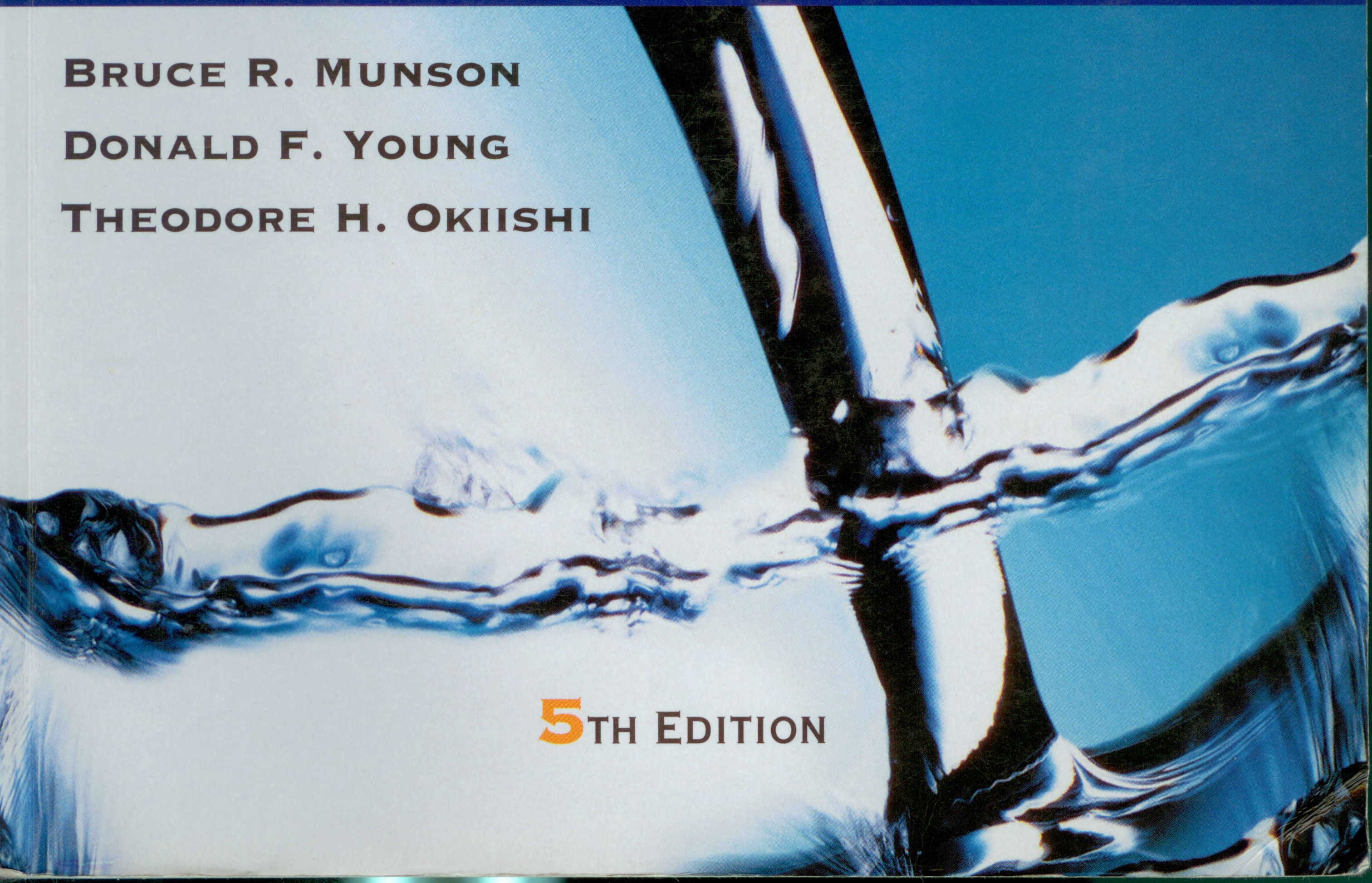
Wiley Asia Student Edition

FUNDAMENTALS OF FLUID MECHANICS

BRUCE R. MUNSON

DONALD F. YOUNG

THEODORE H. OKIISHI



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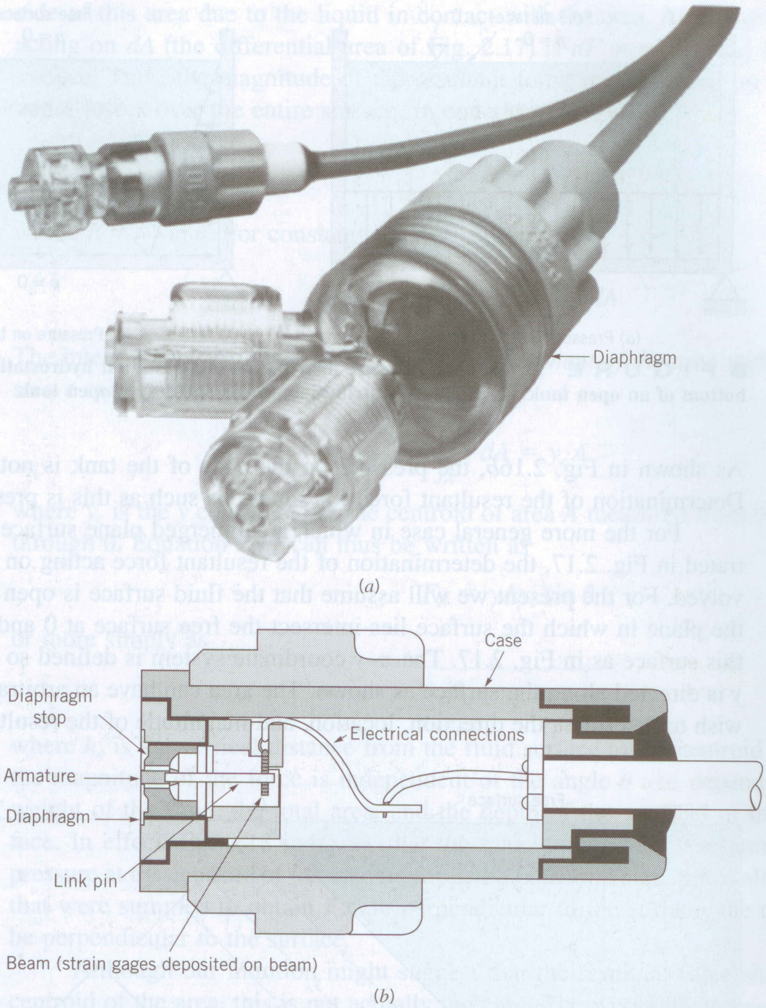


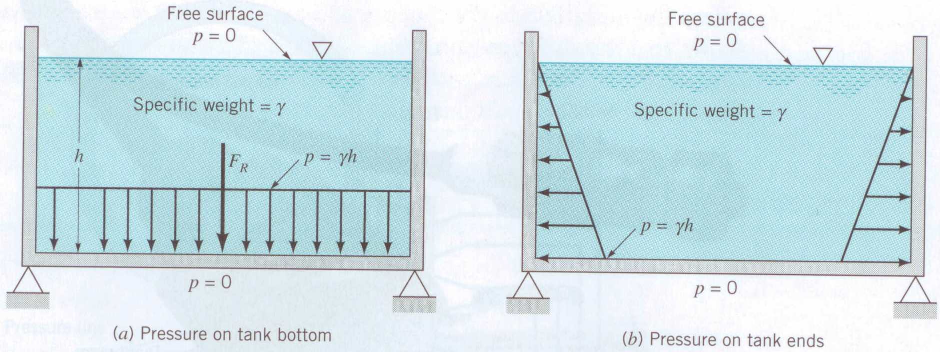
■ FIGURE 2.15 *(a)* **Two different sized strain-gage pressure transducers (Spectramed Models P10EZ and P23XL) commonly used to measure physiological pressures. Plastic domes are filled with fluid and connected to blood vessels through a needle or catheter. (Photograph courtesy of Spectramed, Inc.)** *(b)* **Schematic diagram of the P23XL transducer with the dome removed. Deflection of the diaphragm due to pressure is measured with a silicon beam on which strain gages and an associated bridge circuit have been deposited.**

2.8 Hydrostatic Force on a Plane Surface

V2.3 Hoover dam

When determining the resultant force on an area, the effect of atmospheric pressure often cancels.

When a surface is submerged in a fluid, forces develop on the surface due to the fluid. The determination of these forces is important in the design of storage tanks, ships, dams, and other hydraulic structures. For fluids at rest we know that the force must be *perpendicular* to the surface since there are no shearing stresses present. We also know that the pressure will vary linearly with depth as shown in Fig. 2.16 if the fluid is incompressible. For a horizontal surface, such as the bottom of a liquid-filled tank (Fig. 2.16*a*), the magnitude of the resultant force is simply $F_R = pA$, where p is the uniform pressure on the bottom and A is the area of the bottom. For the open tank shown, $p = \gamma h$. Note that if atmospheric pressure acts on both sides of the bottom, as is illustrated, the *resultant* force on the bottom is simply due to the liquid in the tank. Since the pressure is constant and uniformly distributed over the bottom, the resultant force acts through the centroid of the area as shown in Fig. 2.16*a*.

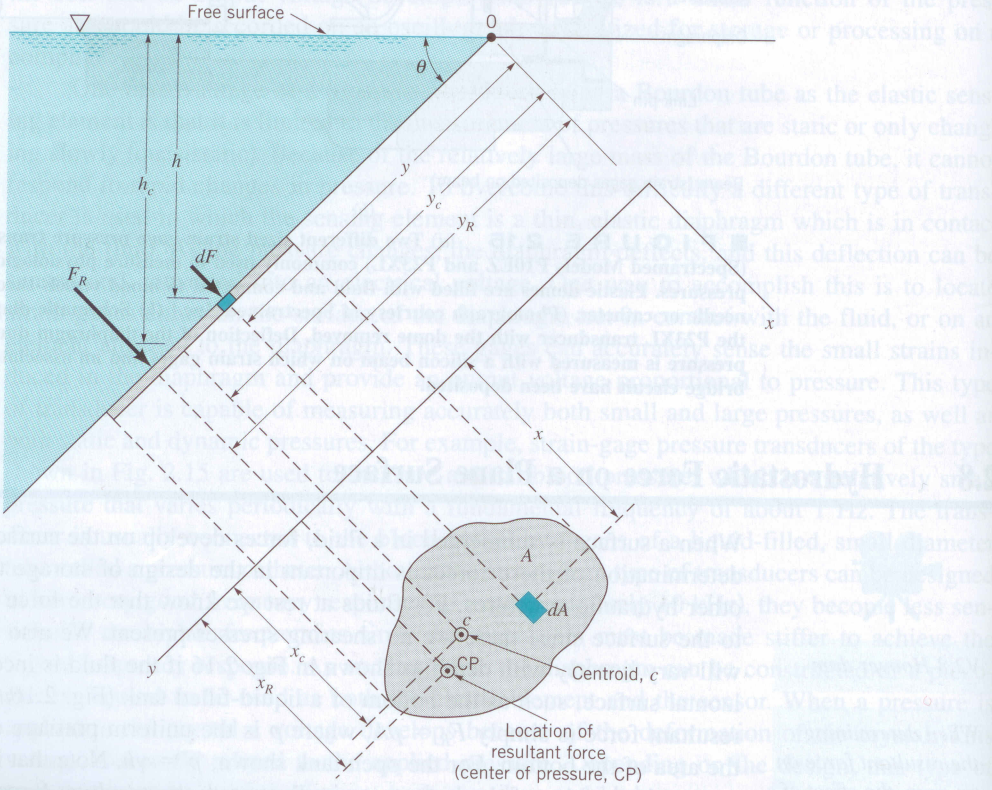


■ **FIGURE 2.16** ***(a)*** **Pressure distribution and resultant hydrostatic force on the bottom of an open tank. *(b)* Pressure distribution on the ends of an open tank.**

The resultant force of a static fluid on a plane surface is due to the hydrostatic pressure distribution on the surface.

As shown in Fig. 2.16*b*, the pressure on the ends of the tank is not uniformly distributed. Determination of the resultant force for situations such as this is presented below.

For the more general case in which a submerged plane surface is inclined, as is illustrated in Fig. 2.17, the determination of the resultant force acting on the surface is more involved. For the present we will assume that the fluid surface is open to the atmosphere. Let the plane in which the surface lies intersect the free surface at 0 and make an angle θ with this surface as in Fig. 2.17. The x–y coordinate system is defined so that 0 is the origin and y is directed along the surface as shown. The area can have an arbitrary shape as shown. We wish to determine the direction, location, and magnitude of the resultant force acting on one



■ **FIGURE 2.17** **Notation for hydrostatic force on an inclined plane surface of arbitrary shape.**

The magnitude of the resultant fluid force is equal to the pressure acting at the centroid of the area multiplied by the total area.

side of this area due to the liquid in contact with the area. At any given depth, h, the force acting on dA (the differential area of Fig. 2.17) is $dF = \gamma h\, dA$ and is perpendicular to the surface. Thus, the magnitude of the resultant force can be found by summing these differential forces over the entire surface. In equation form

$$F_R = \int_A \gamma h\, dA = \int_A \gamma y \sin\theta\, dA$$

where $h = y\sin\theta$. For constant γ and θ

$$F_R = \gamma \sin\theta \int_A y\, dA \tag{2.17}$$

The integral appearing in Eq. 2.17 is the *first moment of the area* with respect to the x axis, so we can write

$$\int_A y\, dA = y_c A$$

where y_c is the y coordinate of the centroid of area A measured from the x axis which passes through 0. Equation 2.17 can thus be written as

$$F_R = \gamma A y_c \sin\theta$$

The magnitude of the resultant fluid force is equal to the pressure acting at the centroid of the area multiplied by the total area.

or more simply as

$$\boxed{F_R = \gamma h_c A} \tag{2.18}$$

where h_c is the vertical distance from the fluid surface to the centroid of the area. Note that the magnitude of the force is independent of the angle θ and depends only on the specific weight of the fluid, the total area, and the depth of the centroid of the area below the surface. In effect, Eq. 2.18 indicates that the magnitude of the resultant force is equal to the pressure at the centroid of the area multiplied by the total area. Since all the differential forces that were summed to obtain F_R are perpendicular to the surface, the resultant F_R must also be perpendicular to the surface.

Although our intuition might suggest that the resultant force should pass through the centroid of the area, this is not actually the case. The y coordinate, y_R, of the resultant force can be determined by summation of moments around the x axis. That is, the moment of the resultant force must equal the moment of the distributed pressure force, or

$$F_R y_R = \int_A y\, dF = \int_A \gamma \sin\theta\, y^2\, dA$$

and, therefore, since $F_R = \gamma A y_c \sin\theta$

$$y_R = \frac{\int_A y^2\, dA}{y_c A}$$

The integral in the numerator is the *second moment of the area (moment of inertia)*, I_x, with respect to an axis formed by the intersection of the plane containing the surface and the free surface (x axis). Thus, we can write

$$y_R = \frac{I_x}{y_c A}$$

Use can now be made of the parallel axis theorem to express I_x as

$$I_x = I_{xc} + A y_c^2$$

where I_{xc} is the second moment of the area with respect to an axis passing through its *centroid* and parallel to the x axis. Thus,

$$y_R = \frac{I_{xc}}{y_c A} + y_c \qquad (2.19)$$

Equation 2.19 clearly shows that the resultant force does not pass through the centroid but is always *below* it, since $I_{xc}/y_c A > 0$.

The x coordinate, x_R, for the resultant force can be determined in a similar manner by summing moments about the y axis. Thus,

The resultant fluid force does not pass through the centroid of the area.

$$F_R x_R = \int_A \gamma \sin\theta \, xy \, dA$$

and, therefore,

$$x_R = \frac{\int_A xy \, dA}{y_c A} = \frac{I_{xy}}{y_c A}$$

where I_{xy} is the product of inertia with respect to the x and y axes. Again, using the parallel axis theorem,[1] we can write

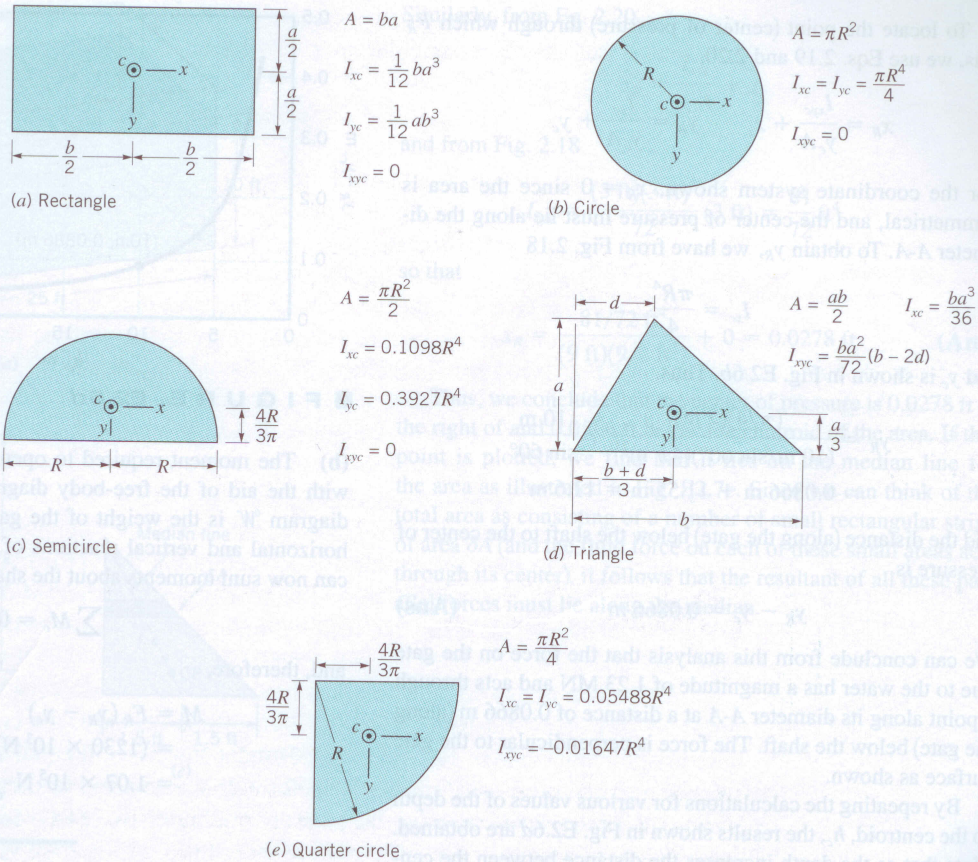
$$x_R = \frac{I_{xyc}}{y_c A} + x_c \qquad (2.20)$$

where I_{xyc} is the product of inertia with respect to an orthogonal coordinate system passing through the *centroid* of the area and formed by a translation of the x–y coordinate system. If the submerged area is symmetrical with respect to an axis passing through the centroid and parallel to either the x or y axes, the resultant force must lie along the line $x = x_c$, since I_{xyc} is identically zero in this case. The point through which the resultant force acts is called the ***center of pressure.*** It is to be noted from Eqs. 2.19 and 2.20 that as y_c increases the center of pressure moves closer to the centroid of the area. Since $y_c = h_c/\sin\theta$, the distance y_c will increase if the depth of submergence, h_c, increases, or, for a given depth, the area is rotated so that the angle, θ, decreases. Centroidal coordinates and moments of inertia for some common areas are given in Fig. 2.18.

F l u i d s i n t h e N e w s

The Three Gorges Dam The Three Gorges Dam being constructed on China's Yangtze River will contain the world's largest hydroelectric power plant when in full operation. The dam is of the concrete gravity type having a length of 2309 meters with a height of 185 meters. The main elements of the project include the dam, two power plants, and navigation facilities consisting of a ship lock and lift. The power plants will contain 26 Francis type turbines, each with a capacity of 700 megawatts. The spillway section, which is the center section of the dam, is 483 meters long with 23 bottom outlets and 22 surface sluice gates. The maximum discharge capacity is 102,500 cubic meters per second. After more than 10 years of construction the dam gates were finally closed, and on June 10, 2003, the reservoir had been filled to its interim level of 135 meters. Due to the large depth of water at the dam and the huge extent of the storage pool, *hydrostatic pressure forces* have been a major factor considered by engineers. When filled to its normal pool level of 175 meters, the total reservoir storage capacity is 39.3 billion cubic meters. The project is scheduled for completion in 2009. (See Problem 2.108.) ■

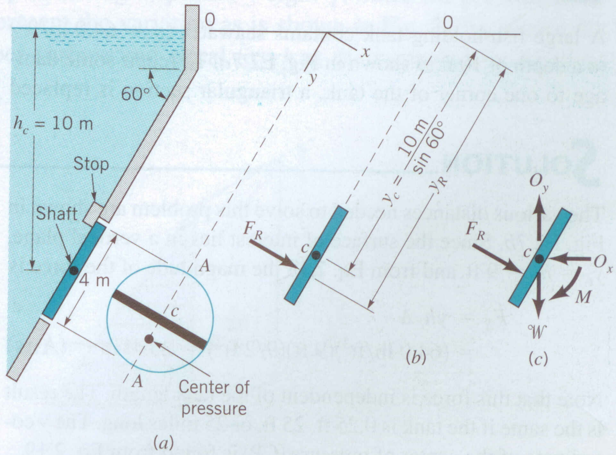
[1] Recall that the parallel axis theorem for the product of inertia of an area states that the product of inertia with respect to an orthogonal set of axes (x–y coordinate system) is equal to the product of inertia with respect to an orthogonal set of axes parallel to the original set and passing through the centroid of the area, plus the product of the area and the x and y coordinates of the centroid of the area. Thus, $I_{xy} = I_{xyc} + Ax_c y_c$.



■ **FIGURE 2.18** Geometric properties of some common shapes.

EXAMPLE 2.6 Hydrostatic Pressure Force on a Plane Rectangular Surface

The 4-m-diameter circular gate of Fig. E2.6*a* is located in the inclined wall of a large reservoir containing water ($\gamma = 9.80\ \text{kN/m}^3$). The gate is mounted on a shaft along its horizontal diameter. For a water depth of $h_c = 10$ m above the shaft determine: (a) the magnitude and location of the resultant force exerted on the gate by the water, and (b) the moment that would have to be applied to the shaft to open the gate.



■ **FIGURE E2.6*a–c***

SOLUTION

(a) To find the magnitude of the force of the water we can apply Eq. 2.18,

$$F_R = \gamma h_c A$$

and since the vertical distance from the fluid surface to the centroid of the area is 10 m it follows that

$$F_R = (9.80 \times 10^3\ \text{N/m}^3)(10\ \text{m})(4\pi\ \text{m}^2)$$
$$= 1230 \times 10^3\ \text{N} = 1.23\ \text{MN} \qquad \textbf{(Ans)}$$

To locate the point (center of pressure) through which F_R acts, we use Eqs. 2.19 and 2.20,

$$x_R = \frac{I_{xyc}}{y_c A} + x_c \qquad y_R = \frac{I_{xc}}{y_c A} + y_c$$

For the coordinate system shown, $x_R = 0$ since the area is symmetrical, and the center of pressure must lie along the diameter *A-A*. To obtain y_R, we have from Fig. 2.18

$$I_{xc} = \frac{\pi R^4}{4}$$

and y_c is shown in Fig. E2.6*b*. Thus,

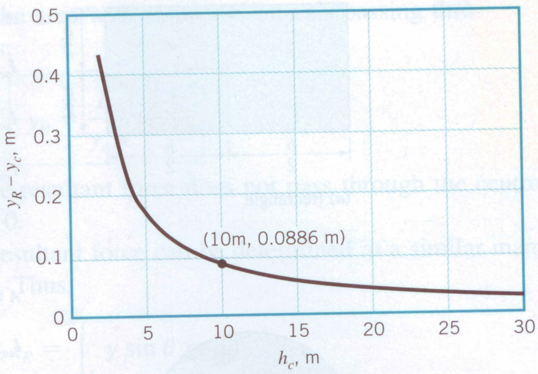
$$y_R = \frac{(\pi/4)(2\text{ m})^4}{(10\text{ m}/\sin 60°)(4\pi\text{ m}^2)} + \frac{10\text{ m}}{\sin 60°}$$
$$= 0.0866\text{ m} + 11.55\text{ m} = 11.6\text{ m}$$

and the distance (along the gate) below the shaft to the center of pressure is

$$y_R - y_c = 0.0866\text{ m} \qquad \textbf{(Ans)}$$

We can conclude from this analysis that the force on the gate due to the water has a magnitude of 1.23 MN and acts through a point along its diameter *A-A* at a distance of 0.0866 m (along the gate) below the shaft. The force is perpendicular to the gate surface as shown.

By repeating the calculations for various values of the depth to the centroid, h_c, the results shown in Fig. E2.6*d* are obtained. Note that as the depth increases the distance between the center of pressure and the centroid decreases.



■ FIGURE E2.6*d*

(b) The moment required to open the gate can be obtained with the aid of the free-body diagram of Fig. E2.6*c*. In this diagram $\mathcal{W}$ is the weight of the gate and O_x and O_y are the horizontal and vertical reactions of the shaft on the gate. We can now sum moments about the shaft

$$\sum M_c = 0$$

and, therefore,

$$M = F_R(y_R - y_c)$$
$$= (1230 \times 10^3\text{ N})(0.0866\text{ m})$$
$$= 1.07 \times 10^5\text{ N}\cdot\text{m} \qquad \textbf{(Ans)}$$

EXAMPLE 2.7 Hydrostatic Pressure Force on a Plane Triangular Surface

A large fish-holding tank contains seawater ($\gamma = 64.0\text{ lb/ft}^3$) to a depth of 10 ft as shown in Fig. E2.7*a*. To repair some damage to one corner of the tank, a triangular section is replaced with a new section as illustrated. Determine the magnitude and location of the force of the seawater on this triangular area.

SOLUTION

The various distances needed to solve this problem are shown in Fig. E2.7*b*. Since the surface of interest lies in a vertical plane, $y_c = h_c = 9$ ft, and from Eq. 2.18 the magnitude of the force is

$$F_R = \gamma h_c A$$
$$= (64.0\text{ lb/ft}^3)(9\text{ ft})(9/2\text{ ft}^2) = 2590\text{ lb} \qquad \textbf{(Ans)}$$

Note that this force is independent of the tank length. The result is the same if the tank is 0.25 ft, 25 ft, or 25 miles long. The *y* coordinate of the center of pressure (CP) is found from Eq. 2.19,

$$y_R = \frac{I_{xc}}{y_c A} + y_c$$

and from Fig. 2.18

$$I_{xc} = \frac{(3\text{ ft})(3\text{ ft})^3}{36} = \frac{81}{36}\text{ ft}^4$$

so that

$$y_R = \frac{81/36\text{ ft}^4}{(9\text{ ft})(9/2\text{ ft}^2)} + 9\text{ ft}$$
$$= 0.0556\text{ ft} + 9\text{ ft} = 9.06\text{ ft} \qquad \textbf{(Ans)}$$

(1)靜力：$I = \int x^2 dA$

Radius of gyration(κ)

$I = m\kappa^2$

平行軸定理

$I_{x'x'} = I_{xx} + md^2$

(2)流力：*Yp=Yc+Ixc/(YcA)*

(3)動力：$\Gamma = I\alpha$

(4)材力：$\sigma = \frac{My}{I}, \sigma = \frac{M_2(I_{yz}z - I_y y)}{I_y I_z - I_{yz}^2}$

$M_{AB} = \frac{2EI}{L}[2\theta_A + \theta_B - 3(\frac{\Delta}{L})] + (FEM)_{AB}$

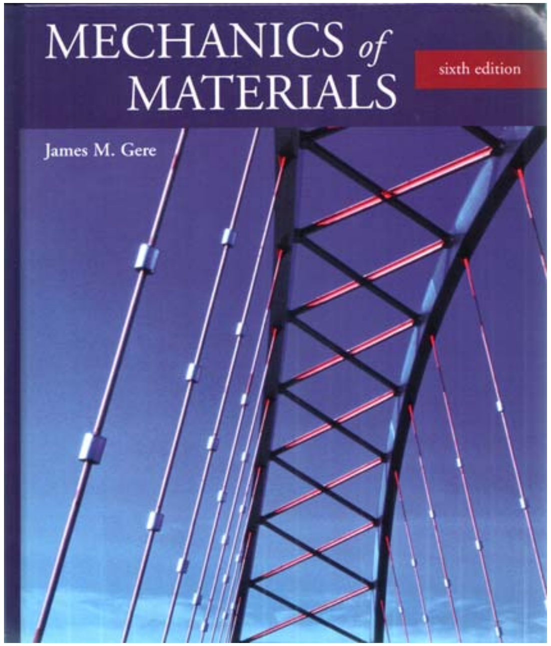
(5)結構：$M_{BA} = \frac{2EI}{L}[2\theta_A + \theta_A - 3(\frac{\Delta}{L})] + (FEM)_{BA}$

$$I_{ij} = \iint_A \left(r^2 \delta_{ij} - x_i x_j\right) dA$$

FIGURES FOR

CHAPTER 12

REVIEW OF CENTROIDS AND MOMENTS OF INERTIA



Click the mouse or use the arrow keys to move to the next page.
Use the ESC key to exit this chapter.

之前曾經說過所謂的應力應變爲二階張量滿足以下的關係式

$$\left[\overline{T}\right] = [L][T][L]^T$$ 二階張量轉換關係

$$[L] = \begin{bmatrix} COS\Theta & SIN\Theta \\ -SIN\Theta & COS\Theta \end{bmatrix}$$ 所謂的矩陣 新一軸舊一軸的夾角依此類推…

因爲這樣說起來很模糊，所以實際利用二階張量轉換的概念，演練一下曾經學過的應力轉換題目

例題;結構表面上某一點存在平面應力的問題，應力大小

$\sigma_x = -46Mpa$ $\sigma_Y = 12Mpa$ $\tau_{xy} = \text{-19Mpa}$ 順時針旋轉 15 度

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} COS15 & SIN15 \\ -SIN15 & COS15 \end{bmatrix} \begin{bmatrix} -46 & -19 \\ -19 & 12 \end{bmatrix} \begin{bmatrix} COS15 & -SIN15 \\ SIN15 & COS15 \end{bmatrix}$$

演算出來的結果和課本所給複雜的公式演算出的結果一樣

藉由二階張量的轉換，可以不用利用材力課本所推出的繁複公式，便可簡單計算出元素再轉任何角度旋轉後的應力變換。這樣一來張量也比較不那麼抽象了

矩　　　　陣

線性代數:

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

利用高斯消去法可求解

例. $$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 4 & 25 & 26 & 9 \\ 2 & 26 & 44 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 51 \\ 94 \\ 171 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 9 & 18 & 9 \\ 0 & 18 & 40 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \\ 82 \\ 171 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 9 & 18 & 9 \\ 0 & 0 & 4 & 16 \\ 0 & 0 & 16 & 80 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \\ 28 \\ 144 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 9 & 18 & 9 \\ 0 & 0 & 4 & 16 \\ 0 & 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \\ 28 \\ 32 \end{bmatrix}$$

$$p = -4 \quad , \quad q = 3 \quad , \quad r = -1 \quad , \quad s = 2$$

矩陣操作

降階: $[\ ]_{4\times4} \times [\ ]_{4\times1} = [\ ]_{4\times1}$

左右通乘 $A_{3\times4}$ ， 且 $A_{4\times3} \times [\ ]_{3\times1} = [\ ]_{4\times1}$

即 $A_{3\times4} \times [\ ]_{4\times4} \times A_{4\times3} \times [\ ]_{3\times1} = A_{3\times4} \times [\ ]_{4\times1}$

$\Rightarrow$ $[\ ]_{3\times3} \times [\ ]_{3\times1} = [\ ]_{3\times1}$

$A_{4\times3}$ 如何獲得

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 4 & 25 & 26 & 9 \\ 2 & 26 & 44 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 51 \\ 94 \\ 171 \end{bmatrix}$$

$\because p + 4q + 2r + 0s = 6$

$\therefore p = (-4)q + (-2)r + 0s + 6$

$$\Rightarrow \quad 令\, a_{4\times3} = \begin{bmatrix} -4 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} -4 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} q \\ r \\ s \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

經過降階原式:

$$\Rightarrow \begin{bmatrix} -4 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 & 0 \\ 4 & 25 & 26 & 9 \\ 2 & 26 & 44 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix}\begin{bmatrix} -4 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 51 \\ 94 \\ 171 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 18 & 9 \\ 18 & 40 & 34 \\ 9 & 34 & 89 \end{bmatrix}\begin{bmatrix} q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 27 \\ 82 \\ 171 \end{bmatrix}$$

HOMEWOREK:

使上三角形為零，並寫出 $b_{4\times3}$ ， $b_{3\times2}$ ， $b_{2\times1}$？

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 4 & 25 & 26 & 9 \\ 2 & 26 & 44 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 51 \\ 94 \\ 171 \end{bmatrix} \qquad 令\, b_{4\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{-9}{89} & \frac{-34}{89} \end{bmatrix}$$

$$b_{3\times4} \times [\ ]_{4\times4} \times b_{4\times3} \times [\ ]_{3\times1} = b_{3\times4} \times [\ ]_{4\times1}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-9}{89} \\ 0 & 0 & 1 & \frac{-34}{89} \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 & 0 \\ 4 & 25 & 26 & 9 \\ 2 & 26 & 44 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{-9}{89} & \frac{-34}{89} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-9}{89} \\ 0 & 0 & 1 & \frac{-34}{89} \end{bmatrix}\begin{bmatrix} 6 \\ 51 \\ 94 \\ 171 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 4 & \frac{2114}{89} & \frac{2008}{89} \\ 2 & \frac{2008}{89} & \frac{2760}{89} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{3000}{89} \\ \frac{2552}{89} \end{bmatrix} \qquad 令\, b_{3\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-178}{2760} & \frac{-2008}{2760} \end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}\frac{601}{690} & \frac{878}{345}\\ \frac{878}{345} & \frac{235672}{30705}\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix}=\begin{bmatrix}\frac{127448}{30705}\\ \frac{394448}{30705}\end{bmatrix}$$

令 $b_{2\times1}=\begin{bmatrix}\frac{-878}{2648}\\ 1\end{bmatrix}$

$$\Rightarrow\left[\frac{276345}{10163355}\right][p]=\left[\frac{-1105380}{10163355}\right]$$

$\Rightarrow$ $p=-4$, $q=3$, $r=-1$, $s=2$

矩陣在土力，材力，結構上之應用

勁度矩陣 $\times$ 位移 = 力

數學式:　　　　物理意義:

$$\begin{bmatrix}5 & -4 & 1 & 0\\ -4 & 6 & -4 & 1\\ 1 & -4 & 6 & -4\\ 0 & 1 & -4 & 5\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}$$

...單位為1之力作用在u_2點上。

$$\Rightarrow\begin{bmatrix}\frac{14}{5} & \frac{-16}{5} & 1\\ \frac{-16}{5} & \frac{29}{5} & -4\\ 1 & -4 & 5\end{bmatrix}\begin{bmatrix}u_2\\ u_3\\ u_4\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}\frac{15}{7} & \frac{-20}{7}\\ \frac{-20}{7} & \frac{65}{14}\end{bmatrix}\begin{bmatrix}u_3\\ u_4\end{bmatrix}=\begin{bmatrix}\frac{8}{7}\\ \frac{-5}{14}\end{bmatrix}$$

...預使u_3，u_4產生同樣之位移，可在u_3，u_4各施$8/7$及$-5/14$之力即可。

$$\Rightarrow\left[\frac{5}{6}\right][u_4]=\left[\frac{7}{6}\right]$$

...預使u_4產生同樣之位移，可在u_4上施$7/6$的力，但其它各點的位移，不一定和原先的情況相同，僅有u_4 之位移可確定。

Direct Shear Box (直剪盒)

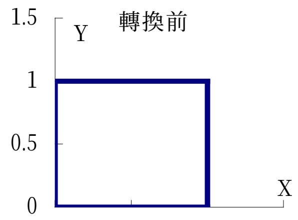
物體受力而產生形變

$\Rightarrow$ $\vec{r}'=[\ ]\times\vec{r}$, 式中之$[\ ]$為轉換矩陣

HOMEWOREK:

令轉換矩陣 $F=\begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ ，求下列圖形之變形。

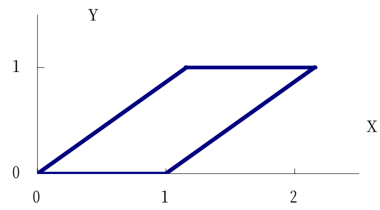
解: 原圖形



$\because \quad X' = FX$

$\Rightarrow X_1' = FX_1 = \begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ， $X_2' = FX_2 = \begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$X_3' = FX_3 = \begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{3}}\\1\\0\end{bmatrix}$ ， $X_4' = FX_4 = \begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1+\frac{2}{\sqrt{3}}\\1\\0\end{bmatrix}$



轉換後之圖形

$A = \begin{bmatrix}\frac{1}{\sqrt{3}}\\1\\0\end{bmatrix}$ ， $A' = \begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}}\\1\\0\end{bmatrix} = \begin{bmatrix}\sqrt{3}\\1\\0\end{bmatrix}$

※角度變成 $30°$ ，長度變成 $\sqrt{3}$ 倍

任一矩陣必可分解為下列情形 ， $F = RU = VR$

其中 R 為旋轉矩陣，UV 為拉伸矩陣

※ R 旋轉矩陣(即僅有方向改變 α 度，長度不變):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\cos^2\alpha - (-\sin^2\alpha) = 1$ 長度不變......R 純粹為使原向量旋轉

若 $y = Rx$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , \quad x^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

$$x^T x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1{}^2 + x_2{}^2 + x_3{}^2 = x \cdot x = |x|^2$$ ......長度的平方

同理轉換後

$y^T y = |y|^2$ 又 $y = Rx$

$\Rightarrow \quad y^T = x^T R^T \quad \Rightarrow \quad |y|^2 = y^T y = (x^T R^T)(Rx) = x^T R^T R x$

$|x|^2 = x^T x$

長度不變 $|y|^2 = |x|^2 \quad \Rightarrow \quad x^T R^T R x = x^T x$

$R^T R = I$ ......即旋轉矩陣之條件

※拉伸矩陣(即僅有長度改變 λ 倍，方向不變):

$x' = Ax = \lambda x \quad \Rightarrow \quad (A - \lambda I)x = 0$

$\Rightarrow \quad \det|A - \lambda I| = 0$

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det|A-\lambda I| = \begin{bmatrix} \frac{\sqrt{3}}{2}-\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}}-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad \lambda_1 = \sqrt{3} \qquad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cos\,60° \\ sin\,60° \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{\sqrt{3}} \qquad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} cos\,150° \\ sin\,150° \\ 0 \end{bmatrix}$$

$$\lambda_3 = 1 \qquad \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

向量分解:　任一向量 x 必可分解作 $\alpha\begin{bmatrix} cos\,60° \\ sin\,60° \\ 0 \end{bmatrix} + \beta\begin{bmatrix} cos\,150° \\ sin\,150° \\ 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

又 $Ax = \lambda x$

$$x' = Ax = \sqrt{3}\alpha\begin{bmatrix} cos\,60° \\ sin\,60° \\ 0 \end{bmatrix} + \frac{1}{\sqrt{3}}\beta\begin{bmatrix} cos\,150° \\ sin\,150° \\ 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

※　$F = RU = VR$ ...I　且　$R^T R = I$

$\Rightarrow$　$F^T = U^T R^T$　...II

II ×I　$\Rightarrow$　得　$F^T F = U^T R^T R U = U^T U$

又令 U 為一對稱矩陣　$\Rightarrow$　$U^T = U$

$\Rightarrow$　$U^T U = UU = U^2 = F^T F$

$U = \sqrt{F^T F}$

$R^T R = I$

同乘 $R^{-1} \Rightarrow R^T R R^{-1} = I R^{-1}$

又　$RR^{-1} = I$　$\Rightarrow$　$R^T = R^{-1}$

$\Rightarrow$　$\because RR^{-1} = I, R^{-1} = R^T$

$\Rightarrow \quad \therefore RR^T = I$

$F = RU = VR$ ...I

$F^T = R^T V^T$ ...II

II ×I $\Rightarrow$ 得 $FF^T = VRR^T V^T = VV^T$

又令 V 為一對稱矩陣 $V^T = V$

$\Rightarrow \quad VV^T = VV = V^2 = FF^T$

$\Rightarrow \quad V = \sqrt{FF^T}$

※$U = \sqrt{F^T F}$ ， $V = \sqrt{FF^T}$

若給定一F 則F^T 也為已知

$U^2 = F^T F = C \quad \Rightarrow \quad U = \sqrt{F^T F}$

求出 U 後 $F = RU \quad \Rightarrow \quad FU^{-1} = R$ 即可得 R

$V^2 = FF^T = \quad \Rightarrow \quad \mathrm{V} = \sqrt{FF^T}$

※$A^{-1} = \dfrac{adj[A]}{\lvert A\rvert}$ ， $adj[A] = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^T$

例. $F = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad F^T = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad U^2 = F^T F = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ \frac{2}{\sqrt{3}} & \frac{7}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \qquad \because A = CDC^{-1}$$

$$\Rightarrow \quad \lambda = \frac{1}{3},1,3 \quad , \quad D = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad , \quad C = \begin{bmatrix} -\sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{3}} & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad C^{-1} = \begin{bmatrix} \frac{-\sqrt{3}}{4} & 0 & \frac{\sqrt{3}}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad U = C\sqrt{D}C^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad U^{-1} = \begin{bmatrix} \frac{5}{2\sqrt{3}} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad R = FU^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad R^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad V = FR^{-1} = \begin{bmatrix} \frac{5}{2\sqrt{3}} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Tensor 張量

Order	0	1	2
	純量	向量	矩陣
	scalar	vector	tensor

慣量 $\begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix}$

$$I_{xx} = \iint y^2 dA$$

$$I_{yy} = \iint x^2 dA$$

$$I_{xy} = -\iint xy\,dA$$

若座標軸旋轉

其不變量:　1. $I_{xx} + I_{yy} = I_{\overline{xx}} + I_{\overline{yy}}$

　　　　　2.行列式值不變…即取 det 相同

證明:　$I_{xx} + I_{yy} = \iint y^2 dA + \iint x^2 dA$

$$= \iint (x^2 + y^2) dA = \iint r^2 dA$$

$$I_{\overline{xx}} + I_{\overline{yy}} = \iint y^2 dA + \iint x^2 dA$$

$$= \iint (x^2 + y^2) dA = \iint r^2 dA$$

$$I_{xx} + I_{yy} = I_{\overline{xx}} + I_{\overline{yy}}$$ 故得證

ill conditioned problems　(病態問題)
有效數字的取決與否

$$\begin{bmatrix} 1 & 1 \\ 1 & -1.014 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{matrix} x = 1.007 \\ y = 0.993 \end{matrix}$$

若四捨五入

$$\begin{bmatrix} 1 & 1 \\ 1 & -1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{matrix} x = 1.005 \\ y = 0.995 \end{matrix}$$ 則兩解相差不多

此為 Well-behaved(健康型)

但若題目為

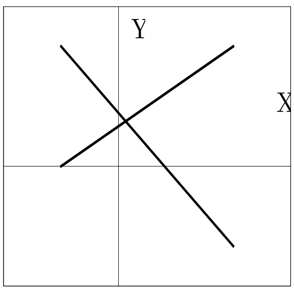
$$\begin{bmatrix} 1 & 1 \\ 1 & 1.014 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{matrix} x = 144.9 \\ y = -142.9 \end{matrix}$$

若四捨五入

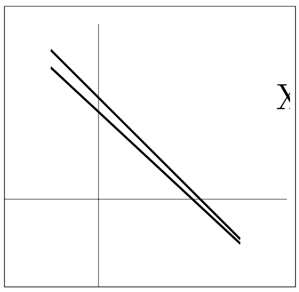
$$\begin{bmatrix} 1 & 1 \\ 1 & 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{matrix} x = 200.2 \\ y = 0 - 200 \end{matrix}$$ 則兩解相差太多

此為 ill-conditioned(病態型)

(健康型)



(病態型)



判定原則

$Ax_1 = \lambda_1 x_1$　　λ_1, λ_2 為 eigenvalue

$Ax_2 = \lambda_2 x_2$　　x_1, x_2 為 eigenvector

$Ax = y$　　$A(x + \Delta x) = (y + \Delta y)$　　若 $\frac{\Delta x}{\Delta y} \rangle\rangle\rangle 1$ 則為病態

依向量分解

$Ax = y = \alpha x_1 + \beta x_2$ 同除 A

$$x = \frac{\alpha}{\lambda_1} x_1 + \frac{\beta}{\lambda_2} x_2 = (\frac{\lambda_2}{\lambda_1} \alpha x_1 + \beta x_2)\frac{1}{\lambda_2}$$

$$A(x + \Delta x) = (y + \Delta y)$$

$$x + \Delta x = \frac{\alpha + \Delta\alpha}{\lambda_1} x_1 + \frac{\beta + \Delta\beta}{\lambda_2} x_2 = (\frac{\lambda_2}{\lambda_1}(\alpha + \Delta\alpha) x_1 + (\beta + \Delta\beta) x_2)\frac{1}{\lambda_2}$$

其中 $\Delta\alpha, \Delta\beta \to 0$ ，故影響不太，其主因在於 $\frac{\lambda_2}{\lambda_1}$

即為 $\frac{|\lambda_m|_{max}}{|\lambda_n|_{min}}$ 其值越接近 1，則越健康，值大則影響越大

$\frac{|\lambda_m|_{max}}{|\lambda_n|_{min}}$ 可代表 eigenvalue 的分散情況

矩陣的應用之範圍

1. 力平衡　$F_1 + F_2 = P$ ，$F_1 L = Pa$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ L & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} P \\ Pa \end{bmatrix}$$

2. Flow Equilibrium
3. Least Square
4. Grapher (View Point)
5. 連體力學變形機制 $F = RU = VR$

※矩　陣　運　算

1. 加法 $[A]_{n\times m} = [B]_{n\times m} + [C]_{n\times m}$
2. 乘法 $[A]_{n\times p} = [B]_{n\times m} \cdot [C]_{m\times p}$

3. 分解 $F = RU = VR$

4. 轉置 Transpose A^T

$$A_{n\times m} \to A_{m\times n} \quad , \quad A_{ij} \to A_{ji}$$

5. 特徵值 eigenvalues

$$Ax = \lambda x \quad \Rightarrow (A-\lambda)x = 0$$

所以給一個A我們可以找到C和D(對稱)

使 $A = CDC^{-1}$ 成立

例.

$$\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ Ax_2 &= \lambda_2 x_2 \\ Ax_3 &= \lambda_3 x_3 \end{aligned} \quad \Rightarrow D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad , \quad C = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}_{3\times 3}$$

$\Rightarrow \quad AC = CD$

$\Rightarrow \quad A = CDC^{-1}$ 即 $f(A) = C \cdot f(D) \cdot C^{-1}$

例. $A^n = (C \cdot D \cdot C^{-1} \cdot C \cdot D \cdot C^{-1} \cdot C \cdot D \cdot C^{-1} \ldots\ldots \cdot C \cdot D \cdot C^{-1})$

又 $C^{-1}C = I \qquad A^n = C \cdot D^n \cdot C^{-1}$

同理 $\sqrt{A} = C\sqrt{D}C^{-1}$ 等等

Degenerate Problem(重根問題；退化問題)

Jordan Form(解重根) $AC = CJ$

$$J = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & c \end{bmatrix}$$

例. $A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$

$\because (A-\lambda I)x=0$

$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 & 3 \\ -1 & 0-\lambda & -3 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad -\lambda^3+6\lambda^2-32=0 \qquad \Rightarrow \lambda=-2,4,4$

$\Rightarrow \quad D=\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad , \quad C=\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

$AC=CD$ 存在，但 $A=CDC^{-1}$ 不存在($\because \det C=0$，C^{-1}不存在)

所以令 $J=\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ 即有重根時多加一個1來彌補。

$$A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}=\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

即 $Ax_1=\lambda_1 x_1$

$Ax_2=\lambda_2 x_2$

$Ax_3=\lambda_2 x_3+x_2$

HOMEWOREK:

求 x_3 及 C^{-1}

$\because (A-\lambda I)x=0$

$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 & 3 \\ -1 & 0-\lambda & -3 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad -\lambda^3+6\lambda^2-32=0 \qquad \Rightarrow \lambda=-2,4,4$

$\Rightarrow \quad \lambda=-2 \qquad x_1=\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$\Rightarrow \quad \lambda=4 \qquad x_1=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$Ax_3=\lambda_2 x_3+x_2$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} t \\ 1-t \\ t-1 \end{bmatrix} \quad \text{令}\, t = 1 \quad \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow C^{-1} = \frac{1}{|C|} adj \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}^T$$

$$\Rightarrow = \frac{1}{-2} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow CJC^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} = A$$

故得證

若 $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow J^2 = \begin{bmatrix} 3^2 & 0 & 0 \\ 0 & 2^2 & 4 \\ 0 & 0 & 2^2 \end{bmatrix}$

$$\Rightarrow J^3 = \begin{bmatrix} 3^3 & 0 & 0 \\ 0 & 2^3 & 12 \\ 0 & 0 & 2^3 \end{bmatrix}$$

….

$$\Rightarrow J^{n-1} = \begin{bmatrix} 3^{n-1} & 0 & 0 \\ 0 & 2^{n-1} & P_{n-1} \\ 0 & 0 & 2^{n-1} \end{bmatrix}$$

$$\Rightarrow \quad J^n = \begin{bmatrix} 3^n & 0 & 0 \\ 0 & 2^n & P_n \\ 0 & 0 & 2^n \end{bmatrix}$$

$\Rightarrow \quad P_n = 2^{n-1} + 2P_{n-1}$

註:

解一. $P_n = 2^{n-1} + 2(2^{n-2} + 2P_{n-2})$

$= 2^{n-1} + 2^{n-1} + 4P_{n-2}$

$= 2^{n-1} + 2^{n-1} + 4(2^{n-3} + 2P_{n-3})$

....

$= 2^{n-1} + 2^1 \times 2^{n-1} + 2^2 \times 2^{n-1} + 2^3 \times 2^{n-1} + \ldots\ldots + 2^{n-1}P_1$

$= (n-1)(2^{n-1}) + 2^{n-1}P_1$

又 $P_1 = 1 \quad \Rightarrow \quad P_n = n \times 2^{n-1}$

解二. $P_n = 2^{n-1} + 2P_{n-1}$

$\Rightarrow \quad P_n - 2P_{n-1} = 2^{n-1}$

$P_n - 2P_{n-1} = 0$ ......補解

$P_n - 2P_{n-1} = 2^{n-1}$ ...特解

令 $P_n = f(n)$, $f(1) = 1$

$P_n - 2P_{n-1} = 0$ 再令 $P = \rho^n$ 代入

$\Rightarrow \rho^n - 2\rho^{n-1} = 0 \quad \Rightarrow \rho - 2 = 0 \quad \Rightarrow \rho = 2$ ......補解

令特解 $= q_n \times 2^{n-1} \quad q_n 2^{n-1} - 2q_{n-1}2^{n-2} = 2^{n-1}$

$q_n - q_{n-1} = 1 \quad \Rightarrow q_n = n + k$ (等差級數)

※ $P_n = k \times 2^n + q_n 2^{n-1}$ 由 $P_1 = 1$ 求出 k 值

解三. 級數解

$P_n = C_n 2^n \quad \Rightarrow P_n - 2P_{n-1} = 2^{n-1}$

$\Rightarrow C_n 2^n - 2C_{n-1}2^{n-1} = 2^{n-1}$

$\Rightarrow C_n - C_{n-1} = \dfrac{1}{2}$ ......成等差

$$\Rightarrow \quad C_n = C_0 + (n-1) \times \frac{1}{2}$$

$$\Rightarrow \quad P_n = C_n 2^n = C_0 2^n + \frac{(n-1)}{2} \times 2^n$$

$$P_1 = 1 \quad \Rightarrow \quad C_0 \times 2^1 + 0 = 1$$

$$\Rightarrow \quad C_0 = \frac{1}{2}$$

$$\Rightarrow \quad C_n = \frac{1}{2} + (n-1) \times \frac{1}{2} = \frac{n}{2}$$

$$\Rightarrow \quad P_n = \frac{n}{2} 2^n = n \cdot 2^{n-1} \text{……特解}$$

故我們可以得知

$$※ J^n = \begin{bmatrix} a & o & o & o & o & o \\ o & b & n \cdot b^{n-1} & o & o & o \\ o & o & b & o & o & o \\ o & o & o & c & n \cdot c^{n-1} & o \\ o & o & o & o & c & n \cdot c^{n-1} \\ o & o & o & o & o & c \end{bmatrix}$$

Eigen Problem

運用於彈簧 $m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1)$

$$m_2 y_2'' = -k_2 (y_2 - y_1)$$

$$\Rightarrow \quad y_1'' = \frac{-(k_1 + k_2)}{m_1} y_1 + \frac{k_2}{m_1} y_2$$

$$y_2'' = \frac{k_1}{m_2} y_1 + \frac{-k_2}{m_2} y_2$$

$$\Rightarrow \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} \frac{-(k_1 + k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

若 $k_1 = 3$ ， $k_2 = 2$ ， $m_1 = m_2 = 1$

$$\Rightarrow \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

令 $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{i\omega t} \quad \Rightarrow \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = -\omega^2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{i\omega t}$

代入原式 $\Rightarrow$ $-\omega^2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}e^{i\omega t}=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}e^{i\omega t}$

$$\Rightarrow \quad -\omega^2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}-5+\omega^2 & 2\\ 2 & -2+\omega^2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=0$$

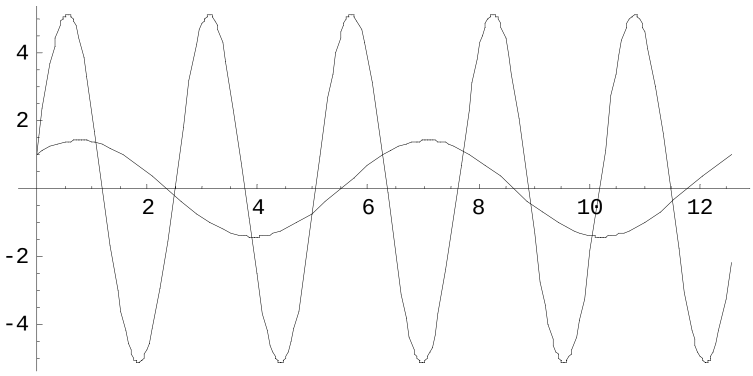
$$\Rightarrow \quad {\omega_1}^2=1 \quad , \quad \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$$

$$\Rightarrow \quad {\omega_2}^2=6 \quad , \quad \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}e^{it}+\begin{bmatrix}2\\ -1\end{bmatrix}e^{i\sqrt{6}t}$$

低頻　高頻

(共振):



HOMEWORK:

三條彈簧

$$m_1 y_1''=-k_1 y_1+k_2(y_2-y_1)$$

$$m_2 y_2''=-k_2(y_2-y_1)+k_3(y_3-y_2)$$

$$m_3 y_3''=-k_3(y_3-y_2)$$

$$\Rightarrow \quad y_1''=\frac{-(k_1+k_2)}{m_1}y_1+\frac{k_2}{m_1}y_2+0y_3$$

$$y_2''=\frac{k_1}{m_2}y_1+\frac{-k_2}{m_2}y_2+\frac{k_3}{m_2}y_3$$

$$y_3''=0y_1+\frac{k_2}{m_3}y_2-\frac{k_3}{m_3}y_3$$

$$\Rightarrow \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_3}{m_3} & \frac{-k_3}{m_3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

餘　式　定　理

HOMEWORK:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ 求 e^A 及 e^{At}

$$det|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = 2,2,3$$

餘式定理: $f(x) = (2-x)^2(3-x)Q(x) + px^2 + qx + r$

運用在矩陣: $f(A) = (2-A)^2(3-A)Q(A) + pA^2 + qA + rI$

微分: $f'(A) = 2(2-A)(-1)(-1)Q'(A) + 2pA + q$

又 $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$e^x = (2-x)^2(3-x)Q(x) + px^2 + qx + r$$

微分 $e^x = 2(2-x)Q'(x) + 2px + q$

$x = 2,2,3$ 代入 $\Rightarrow$
$$p = -2e^2 + e^3$$
$$q = 9e^2 - 4e^3$$
$$r = 4e^3 - 9e^2$$

$$e^A = (2-A)^2(3-A)Q(A) + (-2e^3 + e^3)A^2 + (9e^2 - 4e^3)A + (4e^3 - 9e^2)I$$
$$= \begin{bmatrix} e^2 & 0 & e^3 - e^2 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix}$$

$$e^{At}=\begin{bmatrix} e^{2t} & 0 & e^{3t}-e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

對　稱、反對稱矩陣

$[A]_{n\times m}$　　對稱　$a_{ij}=a_{ji}$　　即　$A^T=A$

例.　$A_{3\times3}=\begin{bmatrix} d & a & b \\ a & e & c \\ b & c & f \end{bmatrix}$

反對稱　$a_{ij}=-a_{ji}$　　即　$A^T=-A$　　且　$i=j$　時　$a_{ij}=0$

例.　$A_{3\times3}=\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

※任意矩陣必可分解為對稱及反對稱矩陣

$C=S+A$ ，　S 為對稱矩陣，A 為反對稱矩陣

$$S=\frac{1}{2}(C+C^T)\quad , \quad A=\frac{1}{2}(C-C^T)$$

證明:

$$S^T=\frac{1}{2}(C+C^T)^T=\frac{1}{2}(C^T+C^{T^T})=\frac{1}{2}(C^T+C)=S$$

$$A^T=\frac{1}{2}(C-C^T)^T=\frac{1}{2}(C^T-C^{T^T})=\frac{1}{2}(C^T-C)=-A$$

故得證

HOMEWOREK:

$v=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$　,　$\omega=\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$　,　$\omega\times v=Av$　求 A 矩陣

$$\Rightarrow \quad \omega\times v=\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ v_1 & v_2 & v_3 \end{vmatrix}=(\omega_2v_3-\omega_3v_2)i+(\omega_3v_1-\omega_1v_3)j+(\omega_1v_2-\omega_2v_1)k$$

$\Rightarrow$ 令 $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $Av=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} av_1+bv_2+cv_3 \\ dv_1+ev_2+fv_3 \\ gv_1+hv_2+iv_3 \end{bmatrix}$

兩式相等

$\Rightarrow \quad a=e=i=0$ ， $b=-\omega_3$ ， $c=\omega_2$ ， $d=\omega_3$

$f=-\omega_1$ ， $g=-\omega_2$ ， $h=\omega_1$

$$\Rightarrow \quad A=\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

HOMEWOREK:

$x'=\omega x$ ， $x_0=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ， $\omega=\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ 且 $\omega_1^2+\omega_2^3+\omega_3^2=1$

試找出一 $x[t]$ ，使 $x'(t)=(\omega_1,\omega_2,\omega_3)\times(x_1,x_2,x_3)$

又 $x(t)=e^{At}x_0$ 證明 $e^{At}(e^{At})^T=I$ 。

$$x'=\omega\times x=Ax$$

$$\Rightarrow \quad A=\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$A^2=\begin{bmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\varpi_1^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{bmatrix}$$

$\Rightarrow \quad det|A-\lambda I|=0 \qquad \Rightarrow \quad \lambda=0,\pm i$

利用餘式定理

$$e^{0t}=0p+0q+r$$
$$e^{it}=(-1)p+iq+r$$
$$e^{-it}=(-1)p+(-i)q+r$$

我們可以得到

$$p=1-(\frac{e^t+e^{-t}}{2})$$
$$q=\frac{e^t-e^{-t}}{2i}$$
$$r=1$$

泰勒展開式:

$$\cos t=1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\frac{1}{6!}t^6+\ldots$$

$$\sin t=t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7+\ldots$$

$$e^{it}=1+\frac{1}{1!}it+\frac{1}{2!}(it)^2+\frac{1}{3!}(it)^3+\ldots$$

$$=1+\frac{1}{1!}it-\frac{1}{2!}t^2-\frac{1}{3!}it^3+\frac{1}{4!}t^4+\frac{1}{5!}it^5-\frac{1}{6!}t^6+\ldots$$

$$
\begin{aligned}
&= \cos t + i\sin t \\
e^{-it} &= 1 + \frac{1}{1!}(-it) + \frac{1}{2!}(-it)^2 + \frac{1}{3!}(-it)^3 + \dots \\
&= 1 - \frac{1}{1!}it - \frac{1}{2!}t^2 + \frac{1}{3!}it^3 + \frac{1}{4!}t^4 - \frac{1}{5!}it^5 - \frac{1}{6!}t^6 + \dots \\
&= \cos t - i\sin t
\end{aligned}
$$

$$
\Rightarrow \quad \begin{aligned} p &= 1-\cos t \\ q &= \sin t \\ r &= 1 \end{aligned}
$$

$$
\Rightarrow \quad e^{At} = (1-\cos t)A^2 + (\sin t)A + I
$$

解一.

$$
e^{At} = \begin{bmatrix} (1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_1\omega_2+\omega_3\sin t & (1-\cos t)\omega_1\omega_3-\omega_2\sin t \\ (1-\cos t)\omega_1\omega_2-\omega_3\sin t & (1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_2\omega_3+\omega_1\sin t \\ (1-\cos t)\omega_1\omega_3+\omega_2\sin t & (1-\cos t)\omega_2\omega_3-\omega_1\sin t & (1-\cos t)(\omega_3^2-1)+1 \end{bmatrix}
$$

$$
(e^{At})^T = \begin{bmatrix} (1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_1\omega_2-\omega_3\sin t & (1-\cos t)\omega_1\omega_3+\omega_2\sin t \\ (1-\cos t)\omega_1\omega_2+\omega_3\sin t & (1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_2\omega_3-\omega_1\sin t \\ (1-\cos t)\omega_1\omega_3-\omega_2\sin t & (1-\cos t)\omega_2\omega_3+\omega_1\sin t & (1-\cos t)(\omega_3^2-1)+1 \end{bmatrix}
$$

$e^{At}(e^{At}) =$

乘開即可得其答案 $e^{At}(e^{At})^T = I$

但過程十分煩雜，計算要小心

解二. $e^{At} = (1-\cos t)A^2 + (\sin t)A + I$

$$
(e^{At})^T = (1-\cos t)(A^2)^T + (\sin t)(A)^T + I^T
$$

$\because A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$ **…為一反對稱矩陣** $A^T = -A$

$\because A^2 = \begin{bmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\varpi_1^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{bmatrix}$ **…為一對稱矩陣** $(A^2)^T = A^2$

故

$$
\begin{aligned}
e^{At}(e^{At})^T &= (1-\cos t)^2A^2(A^2) + (\sin^2 t)AA^T + I^2 + (1-\cos t)(\sin t)A^2A^T + \\
&\quad (1-\cos t)A^2 + (1-\cos t)(\sin t)A(A^2)^T + A\sin t + (1-\cos t)(A^2)^T + A^T\sin t \\
&= (1-\cos t)^2A^4 - (\sin^2 t)A^2 + I^2 - (1-\cos t)(\sin t)A^3 - A^3(1-\cos t)\sin t + A\sin t
\end{aligned}
$$

$$+(1-\cos t)A^2 - A\sin t$$

又 $A^3 + A = 0$　　$A^3 = -A$　　$\Rightarrow$　　$A^4 = -A^2$

$\Rightarrow$　　$e^{At}(e^{At})^T = I$

正　交　矩　陣

正交即代表　$R^T R = I$

$$R_{2\times 2} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$　α為任意值皆成立。

$R_{3\times 3}$:　$R = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$RR^T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$　$a^2 + b^2 + c^2 = 1$

$d^2 + e^2 + f^2 = 1$

$g^2 + h^2 + i^2 = 1$

$ad + be + cf = 0$

$dg + eh + fi = 0$

$ga + hb + ic = 0$

九個未知數，六條方程式 $\Rightarrow$　無限多解

Householder　矩 陣

$$H = I - \frac{2VV^T}{V^T \cdot V}$$　　$V^T \cdot V$ 為內積

鏡射(Mirror)原理:

$Hy = p$　　又　$H(Hy) = y$

$H^2 = I$　即　$HH^T = I$

故任意取一向量V，即可找到一相應的H，且H對稱並正交

令 $V=\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{2}{3}\end{bmatrix} \Rightarrow V^T=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix} \Rightarrow V^T\cdot V=1$

$\Rightarrow \quad VV^T=\begin{bmatrix}\frac{1}{9} & \frac{2}{9} & \frac{2}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\end{bmatrix}$

$\Rightarrow \quad H=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-2\begin{bmatrix}\frac{1}{9} & \frac{2}{9} & \frac{2}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\end{bmatrix}=\begin{bmatrix}\frac{7}{9} & \frac{-4}{9} & \frac{-4}{9}\\ \frac{-4}{9} & \frac{1}{9} & \frac{-8}{9}\\ \frac{-4}{9} & \frac{-8}{9} & \frac{1}{9}\end{bmatrix}$

$HH^T=I$ ， 且 $H=H^T$ 正交且對稱

證明: $H=I-\frac{2VV^T}{V^T\cdot V}$

$H^T=I^T-(\frac{2VV^T}{V^T\cdot V})^T=I-\frac{2(VV^T)^T}{V^T\cdot V}$ 　 $V^T\cdot V$ 內積為純量不影響

$=I-\frac{2}{V^T\cdot V}(V^T)^T V^T$

$=I-\frac{2VV^T}{V^T\cdot V}=H$ 　 對稱

$HH^T=(I-\frac{2VV^T}{V^T\cdot V})(I-\frac{2VV^T}{V^T\cdot V})$

$=I^2-\frac{4VV^T}{V^T\cdot V}+4\frac{VV^TVV^T}{V^T\cdot V\times V^T\cdot V}$

$\because V^TV=V\cdot V=|V|^2$ 　 為長度的平方

原式 $=I^2-\frac{4VV^T}{V^T\cdot V}+\frac{4VV^T}{V^T\cdot V}$

$=I^2=I$ 　 故得證

例: 　 令 $V=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

$H=I-\frac{2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\begin{bmatrix}\alpha & \beta\end{bmatrix}}{\alpha^2+\beta^2}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-2\begin{bmatrix}\frac{\alpha^2}{\alpha^2+\beta^2} & \frac{\alpha\beta}{\alpha^2+\beta^2}\\ \frac{\alpha\beta}{\alpha^2+\beta^2} & \frac{\beta^2}{\alpha^2+\beta^2}\end{bmatrix}$

若 $\alpha^2+\beta^2=1$ 即 V 為一單位向量

$$H=\begin{bmatrix}\beta^2-\alpha^2 & -2\alpha\beta \\ -2\alpha\beta & \alpha^2-\beta^2\end{bmatrix}$$

Homework:

若給一$V=\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$ 且 $\omega_1^2+\omega_2^3+\omega_3^2=1$

求H？ $H=I-\dfrac{2VV^T}{V^T\cdot V}$

$$=I-2\frac{VV^T}{\omega^2{}_1+\omega^2{}_2+\omega^2{}_3}$$

$$=I-2\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}\begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-2\begin{bmatrix}\omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3\\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3\\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2\end{bmatrix}$$

$$=\begin{bmatrix}1-2\omega_1^2 & -2\omega_1\omega_2 & -2\omega_1\omega_3\\ -2\omega_1\omega_2 & 1-2\omega_2^2 & -2\omega_2\omega_3\\ -2\omega_1\omega_3 & -2\omega_2\omega_3 & 1-2\omega_3^2\end{bmatrix}$$

若 $A=\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$

$$e^{At}=\begin{bmatrix}(1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_1\omega_2+\omega_3\sin t & (1-\cos t)\omega_1\omega_3-\omega_2\sin t\\ (1-\cos t)\omega_1\omega_2-\omega_3\sin t & (1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_2\omega_3+\omega_1\sin t\\ (1-\cos t)\omega_1\omega_3+\omega_2\sin t & (1-\cos t)\omega_2\omega_3-\omega_1\sin t & (1-\cos t)(\omega_3^2-1)+1\end{bmatrix}$$

給 $t=\pi$ 時

$$e^{A\pi}=\begin{bmatrix}2\omega_1^2-1 & 2\omega_1\omega_2 & 2\omega_1\omega_3\\ 2\omega_1\omega_2 & 2\omega_2^2-1 & 2\omega_2\omega_3\\ 2\omega_1\omega_3 & 2\omega_2\omega_3 & 2\omega_3^2-1\end{bmatrix}$$

令 $e^{A\pi}=M \quad\Rightarrow\quad -M=H$

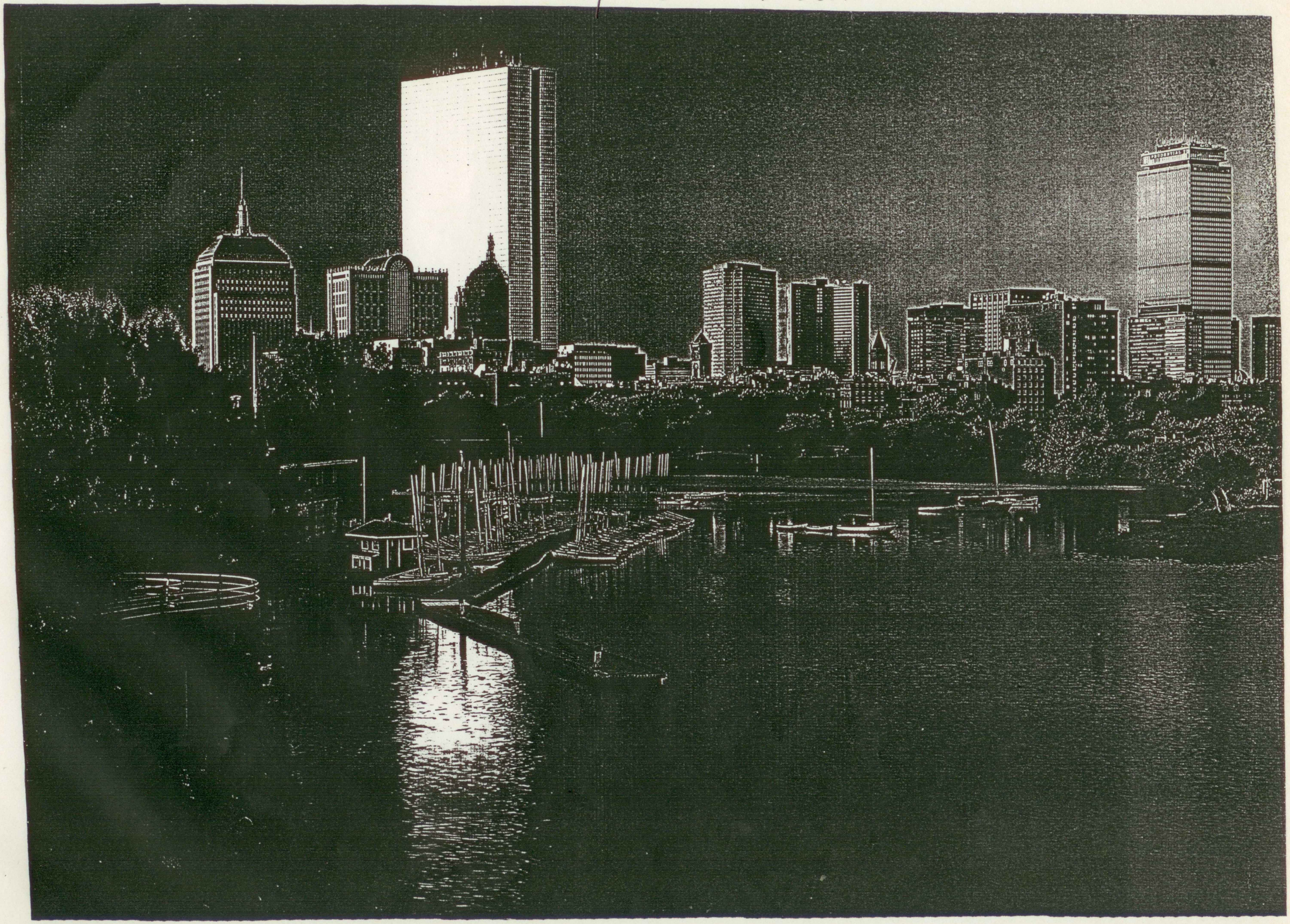
$MM^T=I \quad\Rightarrow\quad (-M)(-M)^T=(-1)(-1)MM^T=MM^T=I$

Hermitian matrix

Real matrix	Complex matrix
a	$a+bi$
$A = A^T \quad \lambda$ *is real*	$\overline{A} = A^T\,(Hermitian) \quad \lambda$ *is real*
$\lvert\underset{\sim}{x}\rvert^2 = \underset{\sim}{x}\cdot\underset{\sim}{x} = x^T\cdot x$	$\lvert\underset{\sim}{x}\rvert^2 = \underset{\sim}{x}\cdot\underset{\sim}{x} = \overline{x}^T\cdot x$
$(\underset{\sim}{x}\cdot\underset{\sim}{y}) = x^T\cdot y$	$(\underset{\sim}{x}\cdot\underset{\sim}{y}) = \overline{x}^T\cdot y$
$A^T\cdot A = I(orthogonal,$正交矩陣$)$	$\overline{A}^T\cdot A = I(unitary,$酉矩陣$)$
$(A\underset{\sim}{x})\cdot\underset{\sim}{y} = \underset{\sim}{x}\cdot(A^T\,\underset{\sim}{y})$	$(A\underset{\sim}{x})\cdot\underset{\sim}{y} = \underset{\sim}{x}\cdot(\overline{A}^T\,\underset{\sim}{y})$
$A=-A^T\,(anti,skew)$ $\lambda=0 \quad \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}$	$-A=\overline{A}^T\,(skew\text{-}Heritian)$ $\begin{bmatrix} ai & d+ei & f+gi \\ -d+ei & bi & h+ji \\ -f+gi & -h+ji & ci \end{bmatrix}$
$\det A = \pm 1$, A *is orthogonal* $\lambda_A = \pm 1$, A *is orthogonal*	$\det U = \pm 1$, U *is unitary* $\lvert\lambda_A\rvert = 1$, U *is unitary*

Hermit03.doc 郭昶志製表 91 年 5 月 15 日

Skyline in Boston



Role of eigen vector for e^{At}

海大河海系　　陳正宗與向天緯

governing equation:

$$\dot{x} = x - 2y$$

$$\dot{y} = 3x - 4y$$

A matrix:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

eigenvalues of A

$$\lambda_1 = -1, \;\; \lambda_2 = -2$$

eigenvectors of A

$$\mathbf{x}_1 = \{1, 1\}, \;\; \mathbf{x}_2 = \{2, 3\}$$

eigenvectors of e^{At}

$$\mathbf{x}_1 = \{1, 1\}, \;\; \mathbf{x}_2 = \{2, 3\}$$

矩陣函數與狀態方程式

海大河海系　　陳正宗

Matrix function:

Continumm mechanics:

$$U = \sqrt{C}$$

Dynamics and control:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{At}\mathbf{f}(t)dt$$

Rigid body dynamics:

$$A = -A^T$$

State equation:

$\dot{\mathbf{x}} = A\mathbf{x}$

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= a_{11}x_1 + \cdots + a_{1n}x_n \\
\frac{dx_2(t)}{dt} &= a_{21}x_1 + \cdots + a_{2n}x_n \\
\frac{dx_3(t)}{dt} &= a_{31}x_1 + \cdots + a_{3n}x_n \\
\cdots &= \cdots + \cdots + \cdots \\
\frac{dx_n(t)}{dt} &= a_{n1}x_1 + \cdots + a_{nn}x_n
\end{aligned}$$

If A is a constant matrix, the solution is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

If A is a function of time, the solution is

$$F(W_1)F(W_2)F(W_3)\cdots F(W_n)\cdots$$

Example 1: Euler-Cauchy equation

Example 2: Bessel equation

Example 3: Lengendre equation

Change to third order ODE, have a 3 by 3 matrix with variable A.

Gauss pivoting by using dummy link and dummy variable

Given $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

(1) Using Gaussian elimination to find $\begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$

Ans

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \xrightarrow{\text{換列}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore \begin{cases} q + r = 0 \\ p + q = 0 \end{cases} \qquad \text{令}\, r = f \qquad \therefore \begin{cases} p = f \\ q = -f \\ r = f \end{cases}$$

$$\therefore \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = f \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$, f is arbitrary

(2) By introducing a dummy variable s such that s=p ,

we have $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \\ s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$. Find $\begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$

Ans

$$\begin{bmatrix}0&1&1\\1&1&0\\1&0&-1\end{bmatrix}\begin{Bmatrix}p\\q\\r\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}\xrightarrow{\text{令}p=s}\begin{bmatrix}1&1&1&-1\\1&1&0&0\\1&0&-1&0\\-1&0&0&1\end{bmatrix}\begin{Bmatrix}p\\q\\r\\s\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&1&1&-1\\0&0&-1&1\\0&-1&-2&1\\0&1&1&0\end{bmatrix}\begin{Bmatrix}p\\q\\r\\s\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}\xrightarrow{\text{令}q=t}\begin{bmatrix}1&1&1&-1&0\\0&1&-1&1&-1\\0&-1&-2&1&0\\0&1&1&0&0\\0&-1&0&0&1\end{bmatrix}\begin{Bmatrix}p\\q\\r\\s\\t\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\end{Bmatrix}$$

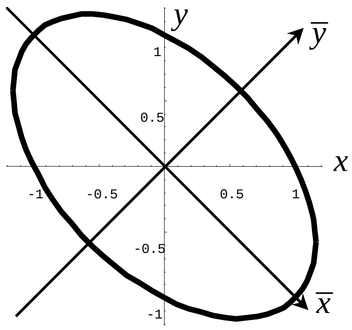
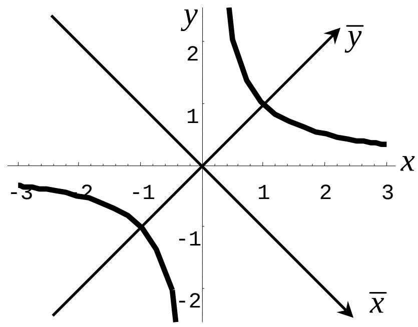
$$\Rightarrow\begin{bmatrix}1&1&1&-1&0\\0&1&-1&1&-1\\0&0&-3&2&-1\\0&0&2&-1&1\\0&0&-1&1&0\end{bmatrix}\begin{Bmatrix}p\\q\\r\\s\\t\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\end{Bmatrix}\Rightarrow\begin{bmatrix}1&1&1&-1&0\\0&1&-1&1&-1\\0&0&-3&2&-1\\0&0&0&\frac{1}{3}&\frac{1}{3}\\0&0&0&\frac{1}{3}&\frac{1}{3}\end{bmatrix}\begin{Bmatrix}p\\q\\r\\s\\t\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\end{Bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&1&1&-1&0\\0&1&-1&1&-1\\0&0&-3&2&-1\\0&0&0&\frac{1}{3}&\frac{1}{3}\\0&0&0&0&0\end{bmatrix}\begin{Bmatrix}p\\q\\r\\s\\t\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\end{Bmatrix}$$

$$\therefore\begin{cases}\frac{1}{3}s+\frac{1}{3}=0\\-3r+2s-t=0\\q-r+s-t=0\\p+q+r-s=0\end{cases}\qquad \text{令}\, s=f \qquad \therefore\begin{cases}p=s=f\\t=q=-f\\r=f\end{cases}$$

$$\therefore\begin{Bmatrix}p\\q\\r\end{Bmatrix}=f\begin{Bmatrix}1\\-1\\1\end{Bmatrix}$$

特徵值與特徵向量在二次式的幾何意義

	在課堂（Jan. 09, 2008）	自我檢定
Original equation	$x^2+xy+y^2=1$	$xy=1$
Quadratic form (16%)	$\{x,y\}\begin{bmatrix}1 & 1/2\\ 1/2 & 1\end{bmatrix}\begin{Bmatrix}x\\ y\end{Bmatrix}=1$	$\{x,y\}\begin{bmatrix}0 & 1/2\\ 1/2 & 0\end{bmatrix}\begin{Bmatrix}x\\ y\end{Bmatrix}=1$
$[A]$ (0%)	$[A]=\begin{bmatrix}1 & 1/2\\ 1/2 & 1\end{bmatrix}$	$[A]=\begin{bmatrix}0 & 1/2\\ 1/2 & 0\end{bmatrix}$
$[A]\underset{\sim}{\phi}_1=\lambda_1\underset{\sim}{\phi}_1$ (12%)	$\lambda_1=\dfrac{1}{2}$ and $\underset{\sim}{\phi}_2=\begin{Bmatrix}\dfrac{1}{\sqrt{2}}\\ -\dfrac{1}{\sqrt{2}}\end{Bmatrix}$	$\lambda_1=-\dfrac{1}{2}$ and $\underset{\sim}{\phi}_1=\begin{Bmatrix}\dfrac{1}{\sqrt{2}}\\ -\dfrac{1}{\sqrt{2}}\end{Bmatrix}$
$[A]\underset{\sim}{\phi}_2=\lambda_2\underset{\sim}{\phi}_2$ (12%)	$\lambda_2=\dfrac{3}{2}$ and $\underset{\sim}{\phi}_2=\begin{Bmatrix}\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\end{Bmatrix}$	$\lambda_2=\dfrac{1}{2}$ and $\underset{\sim}{\phi}_2=\begin{Bmatrix}\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\end{Bmatrix}$
$[A][\Phi]=[\Phi][D]$ (24%)	$[D]=\begin{bmatrix}1/2 & 0\\ 0 & 3/2\end{bmatrix}$ and $[\Phi]=\begin{bmatrix}\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}}\\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\cos(45^\circ) & \sin(45^\circ)\\ -\sin(45^\circ) & \cos(45^\circ)\end{bmatrix}$	$[D]=\begin{bmatrix}-1/2 & 0\\ 0 & 1/2\end{bmatrix}$ and $[\Phi]=\begin{bmatrix}\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}}\\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\cos(45^\circ) & \sin(45^\circ)\\ -\sin(45^\circ) & \cos(45^\circ)\end{bmatrix}$
$\begin{Bmatrix}x\\ y\end{Bmatrix}=[\Phi]\begin{Bmatrix}\bar{x}\\ \bar{y}\end{Bmatrix}$ (0%)	$\begin{Bmatrix}x\\ y\end{Bmatrix}=\begin{bmatrix}\cos(45^\circ) & \sin(45^\circ)\\ -\sin(45^\circ) & \cos(45^\circ)\end{bmatrix}\begin{Bmatrix}\bar{x}\\ \bar{y}\end{Bmatrix}$	$\begin{Bmatrix}x\\ y\end{Bmatrix}=\begin{bmatrix}\cos(45^\circ) & \sin(45^\circ)\\ -\sin(45^\circ) & \cos(45^\circ)\end{bmatrix}\begin{Bmatrix}\bar{x}\\ \bar{y}\end{Bmatrix}$
New system (16%)	$\{\bar{x},\bar{y}\}\begin{bmatrix}1/2 & 0\\ 0 & 3/2\end{bmatrix}\begin{Bmatrix}\bar{x}\\ \bar{y}\end{Bmatrix}=1$	$\{\bar{x},\bar{y}\}\begin{bmatrix}-1/2 & 0\\ 0 & 1/2\end{bmatrix}\begin{Bmatrix}\bar{x}\\ \bar{y}\end{Bmatrix}=1$
New equation (10%)	$\dfrac{1}{2}\bar{x}^2+\dfrac{3}{2}\bar{y}^2=1$	$-\dfrac{1}{2}\bar{x}^2+\dfrac{1}{2}\bar{y}^2=1$
Figure (10%)		

※ λ and ϕ are eigenvalue and eigenvector, respectively.

特徵問題與二次式**.doc**

<table>
<tr><td colspan="3">$$A^{-1}=\frac{1}{\det\left|A\right|}adj(A)$$</td></tr>
<tr><td colspan="3">$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$</td></tr>
<tr><td>$adj(A)$</td><td>$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$</td><td>$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \end{bmatrix}^T$$</td></tr>
</table>

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矩陣函數的新解法——矩陣餘式定理之應用

陳正宗

在線性微分方程式系統中，吾人常遇到如下聯立微分方程式：

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + \cdots\cdots\cdots\cdots\cdots\cdots + a_{2n}x_n$$

$$\frac{dx_3}{dt} = a_{31}x_1 + \cdots\cdots\cdots\cdots\cdots\cdots + a_{3n}x_n$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

而其對應初始條件
$$\begin{cases} \bar{x}_1 = x_1(0) \\ \bar{x}_2 = x_2(0) \\ \vdots \quad\quad \vdots \\ \bar{x}_n = x_n(0) \end{cases}$$

爲方便計，可寫成

$$\frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x} \qquad \underset{\sim}{x}(0) = \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{Bmatrix}$$

其中$\underset{\sim}{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix}$

$A = (a_{ij})_{n\times n}$ 矩陣

上述問題的解析解爲

$$\underset{\sim}{x}(t)=e^{At}\,\underset{\sim}{x}(0)$$

在這裡 e^{At} 就是我們要討論的矩陣函數。關於矩陣函數解法，文獻上 [1] 均以類似矩陣變換 (similar matrix) 原理解之，在此，吾人利用實數矩陣對比餘式定理解之，提供一迅速易懂的解題方法，一獻野人之曝。茲將傳統方法與個人方法分述如下：

一、傳統方法

由文獻上 [1] 可知任一方陣A，可經由特徵值 (eigenvalues) 與特徵向量 (eigenvector) 將其表成類似式如下：

$$AC=CD \qquad (\text{similar form})$$

其中D爲特徵值所組成之對角矩陣

C爲特徵向量組成之矩陣

所以，吾人可將A表爲如下之型式：

$$A=CDC^{-1} \quad \cdots\cdots(1)$$

今考慮A矩陣之函數 f (A)，並將其系列式 (series) 表示如下：

$$f(A)=\sum_{i=0}^{\infty} p_i A^i \qquad (p_i \text{ 爲係數})$$

$$=\sum_{i=0}^{\infty} p_i (CDC^{-1})^i \qquad (\text{將(1)式代入})$$

$$=\sum_{i=0}^{\infty} p_i \overbrace{(CDC^{-1})(CDC^{-1})\cdots\cdots(CDC^{-1})(CDC^{-1})}^{i\text{ 次}}$$

$$=\sum_{i=0}^{\infty} p_i CD^i C^{-1}$$

$$=C\left(\sum_{i=1}^{\infty} p_i D^i\right)C^{-1}$$

又因爲D爲對角矩陣，D^i 相當容易求，答案也就自然算出來，爲說明起見，特舉一例如下：

[問題] 假設 $A=\begin{pmatrix}0 & -2\\ 1 & 3\end{pmatrix}$

求 $f(A)=e^A=?$

[解]：假設 $A\underset{\sim}{x}=\lambda\underset{\sim}{x}$

則 $(A-\lambda I)\underset{\sim}{x}=0$，爲一特徵值問題

解 $\det|A-\lambda I|=0$

可得 $\lambda=1$ or 2

其對應之特徵向量爲

當 $\lambda=1$， $\underset{\sim}{x}=\begin{Bmatrix}2\\ -1\end{Bmatrix}$ $\qquad \lambda=2$， $\underset{\sim}{x}=\begin{Bmatrix}1\\ -1\end{Bmatrix}$

所以 $A[\underset{\sim}{x}_1 \quad \underset{\sim}{x}_2] = [\underset{\sim}{x}_1 \quad \underset{\sim}{x}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

其相似式為，$\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

符合前述(1)式之 $AC = CD$，其中 $C = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$，$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

現在

$$f(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

$$= C\{\sum_{n=0}^{\infty} \frac{1}{n!}(D)^n\} C^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} e^1 & 0 \\ 0 & e^2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e & e^2 \\ -e & -e^2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e - e^2 & 2e - 2e^2 \\ -e + e^2 & -e + 2e^2 \end{pmatrix} \text{ 為其解。}$$

二、矩陣餘式定理的應用解法

在高中時代，我們學過實數系的餘式定理，簡述如下：「一函數 $f(x)$，若除以 $(x - a)$ 則其餘式為 $f(a)$」吾人可寫成下式：

$$f(x) = (x - a)Q(x) + f(a)$$

其中 $Q(x)$ 為商式，$f(a)$ 為餘式，同理當除數為二次式時，其餘式則為一次式，可表成下式

$$f(x) = (ax^2 + bx + c)Q(x) + px + q$$

同理 $Q(x)$ 為商式，$px + q$ 為餘式。

如果吾人將實數 x 改為矩陣，其亦有相同特性。（註：此點尚未見文獻證明，但作者屢試不爽，讀者或可嘗試證之）。

亦即 $f(A) = (aA^2 + bA + cI)Q(A) + pA + qI$

其中 $\begin{cases} a, b, c, p, q \text{ 為常數} \\ I \text{ 為單位矩陣} \end{cases}$

有一點值得注意的乃是實數 1 對應過來是單位矩陣。

利用這點我們將上例重作一次如下：

［問題］已知 $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$，求 $e^A = ?$

解：由一般工程數學的書可知[1]，矩陣本身滿足本身的特性方程式（Characteristic Equation），利用此點可知

$$\begin{vmatrix} 0-\lambda & +2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

其特徵方程式爲 $\lambda^2 - 3\lambda + 2 = 0$ $\therefore\ A^2 - 3A + 2I = 0$

而 $f(A) = (A^2 - 3A + 2I)Q(A) + pA + qI$

經由類比效應（實數 ⇒ 矩陣）可改寫成

$$f(x) = (x^2 - 3x + 2)Q(x) + px + q$$

$\therefore\ e^x = (x^2 - 3x + 2)Q(x) + px + q$

將 $x = 1, 2$ 代入，$\begin{cases} e^1 = p + q \\ e^2 = 2p + q \end{cases}$ 可解得 $\begin{cases} p = e^2 - e \\ q = 2e - e^2 \end{cases}$

將 p，q 代回得

$$\begin{aligned} e^A &= (A^2 - 3A + 2I)Q(A) + (e^2 - e)A + (2e - e^2)I \\ &= (e^2 - e)A + (2e - e^2)I \qquad (\because\ A^2 - 3A + 2I = 0) \\ &= (e^2 - e)\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 2e - e^2 & 0 \\ 0 & 2e - e^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2e^2 + 2e \\ e^2 - e & 3e^2 - 3e \end{pmatrix} + \begin{pmatrix} 2e - e^2 & 0 \\ 0 & 2e - e^2 \end{pmatrix} \\ &= \begin{pmatrix} 2e - e^2 & -2e^2 + 2e \\ e^2 - e & 2e^2 - e \end{pmatrix} \end{aligned}$$

其解和傳統方法相同，驗證本法有其可行性。

針對以上結果，吾人可發覺第二種方法，較具一般性，適合解任意之矩陣函數如：e^A，sinA，$A^{1/2}$……等，其方法均同，另一更大特點乃是當A矩陣特徵值有重根時，第一種方法不易求得C矩陣，而第二種方法則可利用微分運算求得所缺少的方程式，才可求得餘式中的係數，也就是說方程式與未知係數才會一樣多。這在實數系裡，可說成一函數 $f(x)$ 有 $(x-a)^2$ 的因式，則 $f'(x)$（表 $f(x)$ 之微分）必有 $(x-a)$ 的因式，道理是一樣的。基於上述兩個理由，作者建議第二種解法提供同仁參考。讀者可自行找個有重根特徵值的例子，以上述兩種方法進行求解，當可一窺究竟。

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A NEW POINT OF VIEW FOR THE POLAR DECOMPOSITION USING SINGULAR VALUE DECOMPOSITION

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Abstract: In this paper, the singular value decomposition and polar decomposition in continuum mechanics are compared with and the relation is constructed. The matrix analysis is studied and the geometric interpretation is explained. The dual bases can be extracted from the right and left vectors of singular value decomposition. An illustrative example of the simple shear case is shown to see the validity of the proposed formulation.

AMS Subject Classification: 74A05, 15A90
Key Words: continuum mechanics, matrix computation, singular value decomposition

1. Introduction

The polar decomposition theorem in the continuum mechanics can be found in the textbooks [1, 2, 3]. It is well known that the deformation gradient (F) can be decomposed into (VR) or (RU), where R is a rotation matrix, U and V are stretching matrices. The former one

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(VR) can be explained that the total deformation process can be decomposed into rotation first and then stretching, while the latter one (RU) is stretching first and then rotation. In the matrix computation, singular value decomposition (SVD) [4] is a very powerful technique for the matrix decomposition and has been applied to engineering problems successfully [5, 6]. However, the relation between the SVD and the polar decomposition was not discussed before and their geometric interpretations in continuum mechanics were not fully understood to the authors' best knowledge.

In this paper, singular value decomposition technique is employed to understand the deformation mechanism in continuum mechanics. The role of the right and left unitary matrices in the singular value decomposition and their relation to the orthogonal matrix (R) in polar decomposition will be examined. One illustrative example with plane deformation, will be demonstrated to show the deformation mechanism by using the SVD technique. It is shown that the two unitary matrices (Φ and Ψ) in SVD provide dual bases for the deformed and undeformed systems. If the deformed and undeformed infinitesimal elements are expanded according to the dual bases, the transformed coordinates between the deformed and undeformed states can be mapped by a diagonal matrix only.

2. Polar Decomposition and SVD Technique

From the textbooks on continuum mechanics [1, 2, 3], we have

$$F = RU = VR, \tag{1}$$

where F is the deformation gradient matrix, which maps undeformed element $\mathbf{dX}$ to deformed element $\mathbf{dx}$ ($\mathbf{dx} = F\mathbf{dX}$), R is an orthogonal matrix, U and V are positive definite symmetric matrices. The U, V and R matrices can be obtained by

$$U = \sqrt{F^T F}, \tag{2}$$

$$V = \sqrt{F F^T}, \tag{3}$$

$$R = FU^{-1}, \tag{4}$$

where the superscript "T" denotes the transpose of a matrix. By employing the SVD technique [4], the F matrix can be decomposed into

$$F = \Phi\Sigma\Psi^T, \tag{5}$$

where Σ is a diagonal matrix with elements of singular values of F, Φ and Ψ are the right and left unitary matrices, respectively. By substituting Eq. (5) into Eqs.(2) and (3), we obtain

$$U = \Psi\Sigma\Psi^T, \tag{6}$$

$$V = \Phi\Sigma\Phi^T. \tag{7}$$

By substituting Eq.(6) into Eq.(4), we have

$$R = \Phi\Psi^T. \tag{8}$$

According to the property of SVD, we have

$$F\psi_i = \sigma_i\phi_i, \tag{9}$$

$$F^T\phi_i = \sigma_i\psi_i, \tag{10}$$

where σ_i is the ith singular value of F, ϕ_i and ψ_i are the ith column vectors for Φ and Ψ, respectively. According to Eqs.(6) and (7), it is easily found that U and V matrices have the same singular values (eigenvalues) (σ_i) and their eigenvectors are ψ_i and ϕ_i, respectively. If the undeformed element, $\mathbf{dX}$, is expanded in terms of the ψ_i ($i = 1, 2, 3$) bases, we have the new coordinate, $\mathbf{dY}$,

$$\mathbf{dY} = \Psi^T\mathbf{dX}. \tag{11}$$

Similarly, the deformed element, $\mathbf{dx}$, can be expanded in terms of ϕ_i ($i = 1, 2, 3$) bases and the new coordinate for $\mathbf{dy}$ is

$$\mathbf{dy} = \Phi^T\mathbf{dx}. \tag{12}$$

According to $\mathbf{dx} = F\mathbf{dX}$, the formula between the transformed coordinates, $\mathbf{dy}$ and $\mathbf{dY}$, can be derived as

$$\mathbf{dy} = \Sigma\mathbf{dY}. \tag{13}$$

It is found that the two transformed vectors ($\mathbf{dy}$ and $\mathbf{dY}$) can be mapped by the diagonal matrix (Σ) only.

3. An Illustrative Example

Considering a simple shear problem [7] defined by

$$x_1 = X_1 + \frac{2}{\sqrt{3}} X_2, \tag{14}$$

$$x_2 = X_2, \tag{15}$$

$$x_3 = X_3, \tag{16}$$

we have

$$F = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

$$U = \sqrt{F^T F} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

$$R = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-30^o) & -\sin(-30^o) & 0 \\ \sin(-30^o) & \cos(-30^o) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

$$V = FR^{-1} = \begin{bmatrix} \frac{5\sqrt{3}}{6} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

Based on the SVD technique, F can be decomposed into

$$F = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (21)$$

where

$$[\Phi] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(30^o) & -\sin(30^o) & 0 \\ \sin(30^o) & \cos(30^o) & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (22)$$

$$[\Sigma] = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (23)$$

$$[\Psi]^T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(-60^o) & -\sin(-60^o) & 0 \\ \sin(-60^o) & \cos(-60^o) & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (24)$$

By substituting Eqs.(22) $\sim$ (24) into Eqs.(6) $\sim$ (8) and comparing with the results of Eqs.(18) $\sim$ (20), the relations between (U, V, R) and (Φ, Σ, Ψ) in Eqs.(6) $\sim$ (8) are all verified. The dual bases for the undeformed (ψ_1, ψ_2, ψ_3) and deformed states (ϕ_1, ϕ_2, ϕ_3) are shown in Figure 1(a). For the undeformed vector ψ_1, the deformation process $(F = RU)$ can be decomposed into stretching with ratio $\sqrt{3}$ and then rotation -30 degrees as shown in Figure 1(b). A reverse process $(F = VR)$ can be understood that rotation -30 degrees first and then stretching with ratio $\sqrt{3}$ as shown in Figure 1(c). By considering the undeformed vector at the corner of the square as shown in Figure 1(b),

$$\mathbf{dX} = (1, 1, 0)^T \tag{25}$$

we have the transformed vector by using Eq.(11),

$$\mathbf{dY} = (\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, 0)^T. \tag{26}$$

By substituting $\mathbf{dY}$ into Eq.(13), we have

$$\mathbf{dy} = (\frac{3+\sqrt{3}}{2}, \frac{-3+\sqrt{3}}{6}, 0)^T. \tag{27}$$

By substituting $\mathbf{dy}$ in Eq.(27) into Eq.(12), we have

$$\mathbf{dx} = (1 + \frac{2}{\sqrt{3}}, 1, 0)^T, \tag{28}$$

which is exactly the same as $F\mathbf{dX}$. Although the derivation is lengthy, the geometric interpretation in the rotation and stretching stages is clear. Also, the relation of polar decomposition in continuum mechanics and SVD in linear algebra is constructed.

4. Concluding Remarks

The mechanism of deformation can be understood by using the SVD technique instead of polar decomposition in this paper. The relation between the matrices in the SVD and those in the polar decomposition was constructed. Also, the deformation stages of stretching and rotation were clearly interpreted in the shown example for plane deformation. Dual bases for the deformed (ϕ_i) and undeformed (ψ_i) states are imbedded in the two unitary matrices of Φ and Ψ. The transformed coordinates for the deformed state can be mapped into that of the deformed state by a diagonal matrix if the dual bases are adopted.

Acknowledgments

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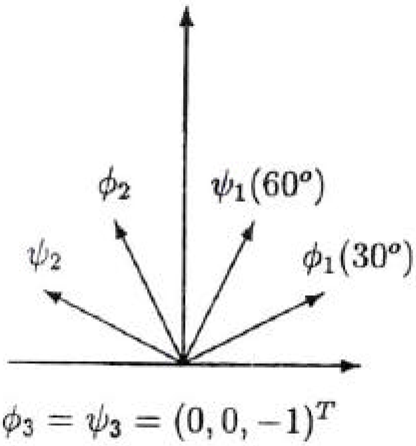

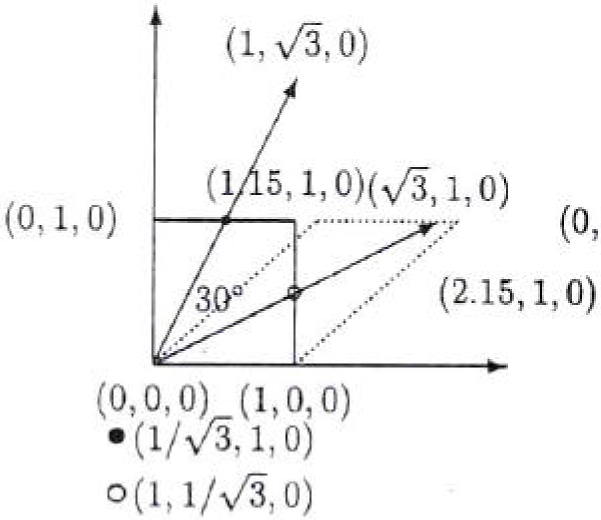

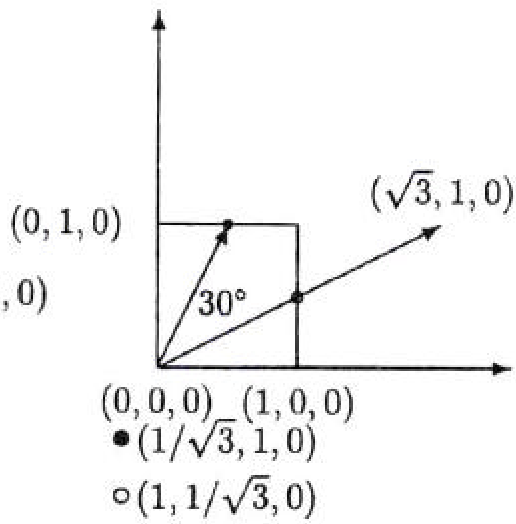

Figure 1(a): The dual bases for the deformed shape and undeformed shape.

Figure 1(b): The undeformed shape (solid line) and deformed shape (dotted line).

Figure 1(c): The undeformed element $(\frac{1}{\sqrt{3}}, 1, 0)$ and deformed element $(1, \frac{1}{\sqrt{3}}, 0)$

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陳正宗　終身特聘教授，生於 1962 年，分別於 1984 年、1986 年、1994 年取得台灣大學土木工程學系學士學位、台灣大學應力所碩士學位及台灣大學土木工程研究所博士學位。1986 至 1990 年間，於中山科學研究院火箭飛彈系統結構部門從事結構力學計算。1994 年至海洋大學河海工程學系擔任副教授一職，1998 年晉升爲教授。2001 與 2004 年分別獲聘海洋大學第一屆優良教師與特聘教授。2005 年獲選台大傑出校友(土木)。2007 年獲聘海洋大學終身特聘教授。2011 年獲華人計算力學會士獎。主要研究領域爲計算力學，曾與洪宏基教授合作推導出對偶積分方程再以對偶邊界元素法求解含退化邊界的邊界值問題作出貢獻。陳正宗教授帶領海大 NTOU/MSV 研究團隊發展出四套對偶邊界元素法程式，Laplace 方程，Helmholtz 方程，修正 Helmholtz 方程與 Navier 方程，並撰寫了兩本有關邊界元素法和有限元素法的中文書籍，也曾受邀到保加利亞（Colloquium of Numerical Analysis, 1996,1997）、阿根廷（WCCM 1998）、奧地利(WCCM 2002)、聖彼得堡（BEM-FEM 2003）、日本京都(ICAM 2007)、中國合肥(ICOME 2009)、香港(ICIP 2010)發表論文演說(Plenary lecture)。連續三次獲得國科會傑出研究獎(1999-2012)及第一屆吳大猷先生紀念獎(2002-2005)並獲聘 A 級計畫主持人(2005-2007)與國科會傑出學者計劃，發表逾百餘篇(168) SCI 論文分佈於 61 種 SCI 期刊並被超過八百餘篇(880)論文引用過。研究論文入榜 ESI 高引用率資料庫。兩篇論文(ASCE, ASME)分別在 SCOPUS 與 WOS 被引超過百次。陳正宗終身特聘教授曾任中國土木水利學刊常務編輯、中國工程學刊土木編委(SCI)、海洋學刊執行編輯與國際計算機材料與連體期刊(Computers, Materials and Continua)編委，國際 Beteq、WCCM、IABEM、Betech、BEM、ICCES、ICCESMM、ICCM、ECOMAS、ICOME、APCOM 執行委員與 MFS-Trefftz 國際會議主席。現爲海洋學刊編委(SCI)，亞太工程學報編委，國際計算方法期刊(Int. J. Comp. Meth.)、國際邊界元素法通訊編委(Boundary Element Communications)、工程科技計算模擬期刊(CMES, SCI)與國際邊界元素法電子期刊(Electronics Journal on BEM)編委，現任工程中邊界元素法期刊編輯(EABE, SCI)與力學期刊(JoM, SCI)副編輯、並審過 73 種期刊論文。由北京清華工程力學系根據最新 ISI Web of Science 資訊統計查得陳正宗終身特聘教授爲二十一世紀世界邊界元素法研究學者 Top 1 。

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