

系級：_____ 學號：_____ 姓名：_____

1. 設 A, B, C 是 n 階方陣。若 $ABC = I$ ，判斷下列各式是否正確。並對正確的式子給與證明。(10%)

- (1) $ACB = I$ (2) $BCA = I$ (3) $CBA = I$ (4) $CAB = I$

2. 試求
$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = ? \quad (10\%)$$

3. $A = \begin{bmatrix} x & 4 & 3 & 7 \\ 0 & 2 & 2 & 5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 & x \\ 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 4 & 1 & 1 \end{bmatrix}$, 又 $\det(AB) = 60$, 試問 $x = ?$ (10%)

4. 已知 $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, 試問 $A^n = ?$ (10%)

5. 已知 $A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$, 試問:

- (1) $\lambda = ?$ 此時 $\text{rank}(A) = 1$ 。(5%)
 (2) $\lambda = ?$ 此時 $\text{rank}(A) = 2$ 。(5%)

6. 已知 $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & a \\ 1 & 1 & 1 \end{bmatrix}$, 向量 $x = \begin{bmatrix} b \\ c \\ 1 \end{bmatrix}$ 為 A 的其中一個特徵值 2 所對應的特徵向

量, 試求 a, b 與 c 為何? (10%)

7. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, 試求 $A^{-1} = ?$ (10%)

8. 利用 Gram-Schmidt 正交化法, 試由向量組 $x^1 = [1 \ 1 \ 0]^T$, $x^2 = [0 \ 1 \ 1]^T$,

$x^3 = [1 \ 0 \ 1]^T$ 建構出一組單位正交向量集 $\{y^1 \ y^2 \ y^3\}$ 。(10%)

9. 已知 $A = \begin{bmatrix} 1 & -2 & -2 \\ 1 & 0 & -3 \\ 1 & -1 & -2 \end{bmatrix}$ ，試問：

- (1) A 的特徵方程為何? (3%)
- (2) 試由 Cayley-Hamilton 定理求 $A^{100} - 3A^{55} = ?$ (6%)
- (3) 若 $A^{-1} = pA^2 + qA + rI$ ，則 $p = ?$, $q = ?$, $r = ?$ (6%)
- (4) 試將 A 化為 Jordan form，即 $A = SJS^{-1}$ (6%)
- (5) 試以 Jordan form 法計算 $A^{100} - 3A^{55} = ?$ (6%)
- (6) 若 $A^{-1} = S\bar{J}S^{-1}$ ，則 $\bar{J} = ?$ (3%)

10. 給方程式 $x_1^2 - 12x_1x_2 + x_2^2 = 70$ ，試以二次式法(quadratic form)將之轉換至主軸，即將舊座標向量 $\mathbf{x}^T = [x_1 \ x_2]$ 轉換至新座標向量 $\mathbf{y}^T = [y_1 \ y_2]$ ，試問：(1) 其轉換矩陣為何? (6%)

(2) 此方程代表何種圓錐曲線? (2%)

(3) $Q = x_1^2 - 12x_1x_2 + x_2^2$ 為何種型式二次式? (2%)
(正定、負定或是不定型)

11. 試解：
$$\begin{cases} y_1'' + 3y_1 - 3y_2 = 10 + 3\sin t \\ y_2'' - 2y_1 + 4y_2 = -10 + 2\sin t \end{cases} \quad \text{且} \quad y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0。$$

(10%)

參考解答:

1. 設 A, B, C 是 n 階方陣。若 $ABC = I$ ，判斷下列各式是否正確。並對正確的式子給與證明。(10%)

$$(1) ACB = I \quad (2) BCA = I \quad (3) CBA = I \quad (4) CAB = I$$

$$ABC = I \Rightarrow BC = A^{-1} \text{ 又 } AA^{-1} = A^{-1}A = I \Rightarrow BCA = I$$

$$ABC = I \Rightarrow AB = C^{-1} \text{ 又 } C^{-1}C = CC^{-1} = I \Rightarrow CAB = I$$

∴ (2)、(4) 為正確

$$2. \text{ 試求 } \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = ? \quad (10\%)$$

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 2 & 3 \\ b^2 & 2b+1 & 2 & 3 \\ c^2 & 2c+1 & 2 & 3 \\ d^2 & 2d+1 & 2 & 3 \end{vmatrix} = 0 \quad (\text{任兩行成比例})$$

$$3. \text{ 已知 } A = \begin{bmatrix} x & 4 & 3 & 7 \\ 0 & 2 & 2 & 5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 & x \\ 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 4 & 1 & 1 \end{bmatrix}, \text{ 又 } \det(A) \times \det(B) = 60, \text{ 試問 } x = ?$$

(10%)

$$\det(A) = \begin{vmatrix} x & 4 & 3 & 7 \\ 0 & 2 & 2 & 5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = 4x+6, \quad \det(B) = \begin{vmatrix} 1 & 0 & 2 & x \\ 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 4 & 1 & 1 \end{vmatrix} = 10$$

$$\det(AB) = \det(A) \times \det(B) = 60 \Rightarrow x = 0$$

4. 已知 $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, 試問 $A^n = ?$ (10%)

$$A^n = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)\lambda^{n-2}}{2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

5. 已知 $A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$, 試問:

(1) $\lambda = ?$ 此時 $\text{rank}(A) = 1$ 。 (5%)

(2) $\lambda = ?$ 此時 $\text{rank}(A) = 2$ 。 (5%)

當 $\text{rank}(A) = 1$ 或 $\text{rank}(A) = 2$ 時, 表示矩陣 A 不滿秩, 即 $\det(A) = 0$

$$\det(A) = (\lambda + 2)(\lambda - 1)^2 = 0 \Rightarrow \lambda = 1 \text{ or } -2$$

當 $\lambda = 1$ 時, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \text{rank}(A) = 1$

當 $\lambda = -2$ 時, $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$

6. 已知 $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & a \\ 1 & 1 & 1 \end{bmatrix}$, 向量 $x = \begin{Bmatrix} b \\ c \\ 1 \end{Bmatrix}$ 為 A 的其中一個特徵值 2 所對應的特徵向量, 試求 a, b 與 c 為何? (10%)

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & a \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} b \\ c \\ 1 \end{Bmatrix} = 2 \begin{Bmatrix} b \\ c \\ 1 \end{Bmatrix} \Rightarrow \begin{cases} b + 2c = 2b & \dots(1) \\ 3b + 4c + a = 2c & \dots(2) \\ b + c + 1 = 2 & \dots(3) \end{cases}$$

由(1)、(3) 可得 $c = \frac{1}{3}$, $b = \frac{2}{3}$ 代回(2)可得 $a = -\frac{8}{3}$

7. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, 試求 $A^{-1} = ?$ (10%)

$\det(A) = 1$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. 利用 Gram-Schmidt 正交化法，試由向量組 $x^1 = [1 \ 1 \ 0]^T$, $x^2 = [0 \ 1 \ 1]^T$,

$x^3 = [1 \ 0 \ 1]^T$ 建構出一組單位正交向量集 $\{y^1 \ y^2 \ y^3\}$ 。(10%)

由 Gram-Schmidt 正交化法，取 $y^1 = \frac{x^1}{|x^1|} = \frac{1}{\sqrt{2}}[1 \ 1 \ 0]^T$

$$v^2 = x^2 - \langle y^1, x^2 \rangle y^1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T \Rightarrow y^2 = \frac{v^2}{|v^2|} = \frac{1}{\sqrt{6}}[-1 \ 1 \ 2]^T$$

$$v^3 = x^3 - \langle y^1, x^3 \rangle y^1 - \langle y^2, x^3 \rangle y^2 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}^T$$

$$\Rightarrow y^3 = \frac{v^3}{|v^3|} = \frac{1}{\sqrt{3}}[1 \ -1 \ 1]^T$$

9. 已知 $A = \begin{bmatrix} 1 & -2 & -2 \\ 1 & 0 & -3 \\ 1 & -1 & -2 \end{bmatrix}$ ，試問：

- (1) A 的特徵方程為何？ (3%)
- (2) 試由 Cayley-Hamilton 定理求 $A^{100} - 3A^{55} = ?$ (6%)
- (3) 若 $A^{-1} = pA^2 + qA + rI$ ，則 $p = ?$ ， $q = ?$ ， $r = ?$ (6%)
- (4) 試將 A 化為 Jordan form，即 $A = PJP^{-1}$ (6%)
- (5) 試以 Jordan form 法計算 $A^{100} - 3A^{55} = ?$ (6%)
- (6) 若 $A^{-1} = P\bar{J}P^{-1}$ ，則 $\bar{J} = ?$ (3%)

$$(1) \det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 & -2 \\ 1 & -\lambda & -3 \\ 1 & -1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 + \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda + 1)^2 = 0$$

$$\Rightarrow \lambda = 1, -1, -1$$

$$(2) \text{由 Cayley-Hamilton 定理可知 } C(\lambda) = \lambda^3 + \lambda^2 - \lambda - 1$$

$$\Rightarrow C(A) = A^3 + A^2 - A - I$$

$$\text{又 } C(1) = 0, C(-1) = 0, C'(-1) = 0$$

$$f(A) = A^{100} - 3A^{55} = Q(A) \cdot C(A) + pA^2 + qA + rI$$

$$f(\lambda) = \lambda^{100} - 3\lambda^{55} = Q(\lambda) \cdot C(\lambda) + p\lambda^2 + q\lambda + r$$

$$\text{由 } \lambda = 1 \text{ 可得 } f(1) = -2 = p + q + r \quad \dots (1)$$

$$\text{由 } \lambda = -1 \text{ 可得 } f(-1) = 4 = p - q + r \quad \dots (2)$$

$$\begin{aligned} \text{微分可得 } f'(\lambda) &= 100\lambda^{99} - 165\lambda^{54} \\ &= Q'(\lambda) \cdot C(\lambda) + Q(\lambda) \cdot C'(\lambda) + 2p\lambda + q \end{aligned}$$

$$\text{由 } \lambda = -1 \text{ 可得 } f'(-1) = -265 = -2p + q \quad \dots (3)$$

$$\text{由 (1)、(2)、(3) 可得 } p = 131, q = -3, r = -130$$

$$\therefore f(A) = 131A^2 - 3A - 130I = \begin{bmatrix} -526 & 6 & 1054 \\ -265 & 1 & 533 \\ -265 & 3 & 531 \end{bmatrix}$$

$$(3) A^3 + A^2 - A - I = 0 \Rightarrow A^{-1}(A^3 + A^2 - A - I) = 0$$

$$\Rightarrow A^{-1} = A^2 + A - I$$

$$\therefore p = 1, q = 1, r = -1$$

$$A^{-1} = A^2 + A - I = \begin{bmatrix} -3 & -2 & 6 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$(4) \text{ 當 } \lambda_1 = 1 \Rightarrow (A - \lambda_1 I)x = \begin{bmatrix} 0 & -2 & -2 \\ 1 & -1 & -3 \\ 1 & -1 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow x^1 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix}$$

$$\text{當 } \lambda_2 = \lambda_3 = -1 \Rightarrow (A - \lambda_2 I)x = \begin{bmatrix} 2 & -2 & -2 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow x^2 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix}$$

因為 2 個特徵值只對應到一個特徵向量，故須引入一條廣義特徵向量

$$Ax^3 = \lambda_2 x^3 + x^2 \Rightarrow (A - \lambda_3 I)x^3 = \begin{bmatrix} 2 & -2 & -2 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow x^3 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} t + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{取 } t = 0, \text{ 可得 } x^3 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore P = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

$$A = PJP^{-1} \Rightarrow \begin{bmatrix} 1 & -2 & -2 \\ 1 & 0 & -3 \\ 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

$$(5) A^{100} - 3A^{55} = PJ_1P^{-1} \Rightarrow J_1 = \begin{bmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_2) & f'(\lambda_2) \\ 0 & 0 & f(\lambda_2) \end{bmatrix}$$

$$\Rightarrow J_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & -265 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{100} - 3A^{55} = PJ_1P^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & -265 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -526 & 6 & 1054 \\ -265 & 1 & 533 \\ -265 & 3 & 531 \end{bmatrix}$$

$$(6) A^{-1} = S\bar{J}S^{-1} \Rightarrow \bar{J} = \begin{bmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_2) & f'(\lambda_2) \\ 0 & 0 & f(\lambda_2) \end{bmatrix} \Rightarrow \bar{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^{-1} = S\bar{J}S^{-1} \Rightarrow A^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 6 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

10. 給方程式 $x_1^2 - 12x_1x_2 + x_2^2 = 70$ ，試以二次式法(quadratic form)將之轉換至主軸，即將舊座標向量 $\mathbf{x}^T = [x_1 \ x_2]$ 轉換至新座標向量 $\mathbf{y}^T = [y_1 \ y_2]$ ，試問：(1) 其轉換矩陣為何？(6%)
- (2) 此方程代表何種圓錐曲線？(2%)
- (3) $Q = x_1^2 - 12x_1x_2 + x_2^2$ 為何種型式二次式？(2%)
(正定、負定或是不定型)

$$(1) x_1^2 - 12x_1x_2 + x_2^2 = 70 \Rightarrow \mathbf{x}^T \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix} \mathbf{x} = 70 \quad \text{其中 } \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\text{由 } |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -6 \\ -6 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 7 \text{ or } -5$$

$$\text{當 } \lambda = 7 \text{ 時，} \begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow x^1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\text{當 } \lambda = -5 \text{ 時，} \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow x^1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\therefore s = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow s^{-1} = s^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\text{令 } \mathbf{y} = s^T \mathbf{x}$$

$$\mathbf{x}^T \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix} \mathbf{x} = 70 \Rightarrow \mathbf{x}^T s \begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix} s^T \mathbf{x} = 70$$

$$\begin{aligned} &\Rightarrow y^T \begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix} y = 70 \\ &\Rightarrow 7y_1^2 - 5y_2^2 = 70 \\ &\Rightarrow \frac{y_1^2}{(\sqrt{10})^2} - \frac{y_2^2}{(\sqrt{14})^2} = 1 \end{aligned}$$

(2) 此為雙曲線

(3) $\because \lambda_1 > 0, \lambda_2 < 0$

$\therefore Q = x_1^2 - 12x_1x_2 + x_2^2$ 為不定型二次式

11. 試解: $\begin{cases} y_1'' + 3y_1 - 3y_2 = 10 + 3\sin t \\ y_2'' - 2y_1 + 4y_2 = -10 + 2\sin t \end{cases}$ 且 $y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0$ 。

(10%)

$$\begin{aligned} \begin{cases} y_1'' + 3y_1 - 3y_2 = 10 + 3\sin t \\ y_2'' - 2y_1 + 4y_2 = -10 + 2\sin t \end{cases} &\Rightarrow \begin{cases} y_1'' \\ y_2'' \end{cases} = \begin{bmatrix} -3 & 3 \\ 2 & -4 \end{bmatrix} \begin{cases} y_1 \\ y_2 \end{cases} + \begin{cases} 10 + 3\sin t \\ -10 + 2\sin t \end{cases} \\ &\Rightarrow y'' = Ax + z \end{aligned}$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -3 - \lambda & 3 \\ 2 & -4 - \lambda \end{vmatrix} = \lambda^2 + 7\lambda + 6 = 0 \Rightarrow \lambda = -1 \text{ or } -6$$

$$\text{當 } \lambda = -1 \Rightarrow \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \begin{cases} x_1 \\ x_2 \end{cases} = \begin{cases} 0 \\ 0 \end{cases} \Rightarrow x^1 = \begin{cases} x_1 \\ x_2 \end{cases} = \begin{cases} 3 \\ 2 \end{cases}$$

$$\text{當 } \lambda = -6 \Rightarrow \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{cases} x_1 \\ x_2 \end{cases} = \begin{cases} 0 \\ 0 \end{cases} \Rightarrow x^2 = \begin{cases} x_1 \\ x_2 \end{cases} = \begin{cases} -1 \\ 1 \end{cases}$$

$$\therefore D = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}, S = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow S^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$\therefore A = SDS^{-1}$$

$$\text{令 } y = S\bar{y} \Rightarrow S \frac{d^2\bar{y}}{dt^2} = AS\bar{y} + z \Rightarrow \frac{d^2\bar{y}}{dt^2} = S^{-1}AS\bar{y} + S^{-1}z$$

$$\therefore \begin{cases} \bar{y}_1'' \\ \bar{y}_2'' \end{cases} = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \begin{cases} \bar{y}_1 \\ \bar{y}_2 \end{cases} + \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{cases} 10 + 3\sin t \\ -10 + 2\sin t \end{cases}$$

$$\Rightarrow \begin{cases} \bar{y}_1'' = -\bar{y}_1 + \sin t \\ \bar{y}_2'' = -6\bar{y}_2 - 10 \end{cases} \Rightarrow \begin{cases} \bar{y}_1'' + \bar{y}_1 = \sin t \\ \bar{y}_2'' + 6\bar{y}_2 = -10 \end{cases}$$

$$\Rightarrow \begin{cases} \bar{y}_1 = c_1 \cos t + c_2 \sin t - \frac{1}{2}t \cos t \\ \bar{y}_2 = c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t - \frac{5}{3} \end{cases}$$

$$\begin{aligned} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} &= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{Bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{Bmatrix} \\ \Rightarrow \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} &= \begin{Bmatrix} 3\bar{y}_1 - \bar{y}_2 \\ 2\bar{y}_1 + \bar{y}_2 \end{Bmatrix} = \begin{Bmatrix} 3c_1 \cos t + 3c_2 \sin t - \frac{3}{2}t \cos t - c_3 \cos \sqrt{6}t - c_4 \sin \sqrt{6}t + \frac{5}{3} \\ 2c_1 \cos t + 2c_2 \sin t - t \cos t + c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t - \frac{5}{3} \end{Bmatrix} \\ \begin{Bmatrix} y_1(0) \\ y_2(0) \end{Bmatrix} &= \begin{Bmatrix} 3c_1 - c_3 + \frac{5}{3} \\ 2c_1 + c_3 - \frac{5}{3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow c_1 = 0, c_3 = \frac{5}{3} \\ \begin{Bmatrix} y_1'(0) \\ y_2'(0) \end{Bmatrix} &= \begin{Bmatrix} 3c_2 - \frac{3}{2} - \sqrt{6}c_4 \\ 2c_2 - 1 + \sqrt{6}c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow c_2 = \frac{1}{2}, c_4 = 0 \\ \therefore \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} &= \begin{Bmatrix} \frac{3}{2} \sin t - \frac{3}{2}t \cos t - \frac{5}{3} \cos \sqrt{6}t + \frac{5}{3} \\ \sin t - t \cos t + \frac{5}{3} \cos \sqrt{6}t - \frac{5}{3} \end{Bmatrix} \end{aligned}$$