

系級：_____ 學號：_____ 姓名：_____

1. 如果地球的人口數在 1970 年為 35 億且以每年 2% 的比率增加，請問何時人口數可到達 500 億。 $(\ln 7 = 1.9459, \ln 10 = 2.3026)$ (10%)

2. 試以分離變數法求解下述微分方程式：

(1) $\frac{dy}{dx} = \frac{y}{x} \ln x$ (8%)

(2) $\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy}$ (8%)

(3) $\frac{dy}{dx} = \frac{2x - 6y + 3}{x - 3y + 1}$ (8%)

3. 已知微分方程式為 $\frac{dy}{dx} + \frac{2x+1}{x} = e^{-2x}$

(1) 此微分方程式為線性或非線性? (2%) 並以一階線性法求解。(8%)

(若為線性，直接求解；若非線性，則轉換成線性，再求解)

(2) 此微分方程式為正合(exact)或非正合? (2%) 並以正合法求解。(8%)

(若正合，直接求解；若非正合，先求出積分因子，再求解)

4. 已知微分方程式為 $y' - \frac{3}{x}y = x^4 y^{\frac{1}{3}}$

(1) 此為何種類型之微分方程式? (Clairaut、Bernoulli 或是 Riccati) (2%)

(2) 試求此微分方程式之解 $y(x) = ?$ (8%)

5. 已知微分方程式為 $\frac{dy}{dx} = -(1+x+x^2) - (2x+1)y - y^2$

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6. 已知微分方程式為 $y = xy' + (y')^3$

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7. 試解下列各微分方程

(1) $y' + \frac{2x \sin y + y^3 e^x}{x^2 \cos y + 3y^2 e^x} = 0$ (8%)

(2) $x^2(y')^2 + 4xyy' + 3y^2 = 0$ (8%)

<參考解答>

1. 如果地球的人口數在 1970 年為 35 億且以每年 2% 的比率增加，請問何時人口數可到達 500 億。 ($\ln 7 = 1.9459$, $\ln 10 = 2.3026$) (10%)

如果 y 為人口數，則可知其數學模式為 $\frac{dy}{dt} = ky = 0.02y$

$$\therefore \frac{dy}{y} = 0.02dt \Rightarrow y(t) = Ce^{0.02t}$$

$$\text{又 } y(0) = 35 \Rightarrow C = 35 \Rightarrow y(t) = 35e^{0.02t}$$

$$\text{又 } y(k) = 500 \Rightarrow 500 = 35e^{0.02k} \Rightarrow e^{0.02k} = \frac{500}{35} = \frac{100}{7}$$

$$\Rightarrow k = \frac{\ln 100 - \ln 7}{0.02} = \frac{2 \cdot 2.3026 - 1.9459}{0.02} \approx 133$$

大約經過 133 年後，即 2103 年時，人口可達 500 億。

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(2) $\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy}$ (8%)

(3) $\frac{dy}{dx} = \frac{2x - 6y + 3}{x - 3y + 1}$ (8%)

$$(1) \frac{dy}{dx} = \frac{y}{x} \ln x \Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x} \ln x dx \Rightarrow \ln|y| = \frac{1}{x} \ln|\ln x| + \ln C$$
$$\Rightarrow y = C \ln x$$

(2) $\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy} \longrightarrow$ 齊次型 ODE

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y} - 1 - \frac{y}{x}$$

$$\text{令 } u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = x \frac{du}{dx} + u \text{ 代回 ODE}$$

$$\text{可得 } \Rightarrow x \frac{du}{dx} + u = \frac{1}{u} - 1 + u \Rightarrow x \frac{du}{dx} = \frac{1-u}{u}$$

$$\Rightarrow -\int \frac{u}{u-1} du = \int \frac{1}{x} dx$$

$$\Rightarrow -(u + \ln|u-1|) = \ln|x| - \ln C$$

$$\Rightarrow e^u \cdot x(u-1) = C$$

$$\Rightarrow (y-x)e^{\frac{y}{x}} = C$$

$$(3) \frac{dy}{dx} = \frac{2x-6y+3}{x-3y+1}$$

$$\text{令 } u = x-3y \Rightarrow \frac{du}{dx} = 1-3\frac{dy}{dx} \quad \text{代回 ODE}$$

$$\text{可得 } \frac{1}{3}\left(1-\frac{du}{dx}\right) = \frac{2u+3}{u+1} \Rightarrow \frac{du}{dx} = 1 - \frac{6u+9}{u+1} = -\frac{5u+8}{u+1}$$

$$\Rightarrow \int \frac{u+1}{5u+8} du = -\int dx$$

$$\Rightarrow \frac{1}{5}\left(u - \frac{3}{5} \ln \left|u + \frac{8}{5}\right|\right) = -x + C$$

$$\Rightarrow \frac{1}{5}(x-3y) - \frac{3}{25} \ln \left|x-3y + \frac{8}{5}\right| + x = C$$

$$\text{or } \frac{2}{5}(x-3y) - \frac{1}{25} \ln \left|x-3y + \frac{8}{5}\right| + y = C$$

3. 已知微分方程式為 $\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x}$

(1) 此微分方程式為線性或非線性? (2%) 並以一階線性法求解。(8%)

(若為線性，直接求解；若非線性，則轉換成線性，再求解)

(2) 此微分方程式為正合(exact)或非正合? (2%) 並以正合法求解。(8%)

(若正合，直接求解；若非正合，先求出積分因子，再求解)

(1) $\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x}$ 此為一階線性微分方程式

$$\text{積分因子 } \mu = e^{\int p(x)dx} = e^{\int \frac{2x+1}{x}dx} = e^{\int \left(2+\frac{1}{x}\right)dx} = e^{2x+\ln x} = x \cdot e^{2x}$$

$$\text{同乘積分因子後可得 } x \cdot e^{2x} \frac{dy}{dx} + e^{2x}(2x+1)y = x \Rightarrow \frac{d}{dx}(x \cdot e^{2x}y) = x$$

$$\Rightarrow x \cdot e^{2x}y = \frac{1}{2}x^2 + C$$

$$\Rightarrow x \cdot e^{2x}y - \frac{1}{2}x^2 = C$$

(2) $\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x} \Rightarrow [(2x+1)y - xe^{-2x}]dx + xdy = 0$

$$\text{令 } M = (2x+1)y - xe^{-2x} \Rightarrow \frac{\partial M}{\partial y} = 2x+1$$

$$N = x \Rightarrow \frac{\partial N}{\partial x} = 1$$

由判斷式 $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ 可知，此為非正合 ODE

由 $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = 2$ 可知 μ 為 $\mu(x)$

$$\therefore \frac{1}{\mu} d\mu = \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) dx \Rightarrow \int \frac{1}{\mu} d\mu = \int \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) dx$$

$$\Rightarrow \ln|\mu| = \int 2 dx = 2x$$

$$\Rightarrow \mu = e^{2x}$$

同乘積分因子後可得 $e^{2x}[(2x+1)y - xe^{-2x}]dx + xe^{2x}dy = 0$ ，此為正合 ODE

$$\therefore \text{可知 } \bar{M} = \frac{\partial \phi}{\partial x} = (2x+1)e^{2x}y - x \Rightarrow \phi = xe^{2x} - \frac{1}{2}x^2 + f(y)$$

$$\bar{N} = \frac{\partial \phi}{\partial y} = xe^{2x} \Rightarrow \phi = xe^{2x}y + g(x)$$

$$\text{比較後可得 } \phi(x, y) = xe^{2x}y - \frac{1}{2}x^2 = C$$

4. 已知微分方程式為 $y' - \frac{3}{x}y = x^4 y^{\frac{1}{3}}$

(1) 此為何種類型之微分方程式? (Clairaut、Bernoulli 或是 Riccati) (2%)

(2) 試求此微分方程式之解 $y(x) = ?$ (8%)

(1) $y' - \frac{3}{x}y = x^4 y^{\frac{1}{3}}$ 此為 Bernoulli ODE

(2) $y' - \frac{3}{x}y = x^4 y^{\frac{1}{3}} \Rightarrow y^{-\frac{1}{3}}y' - \frac{3}{x}y^{\frac{2}{3}} = x^4$

令 $u = y^{\frac{2}{3}} \Rightarrow u' = \frac{2}{3}y^{-\frac{1}{3}}y'$ 代回 ODE 可得

$$\frac{3}{2}u' - \frac{3}{x}u = x^4 \Rightarrow u' - \frac{2}{x}u = \frac{2}{3}x^4 \rightarrow \text{此為一階線性 ODE}$$

積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$

同乘積分因子後可得 $\Rightarrow \frac{1}{x^2}u' - \frac{2}{x^3}u = \frac{2}{3}x^2$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}u\right) = \frac{2}{3}x^2$$

$$\Rightarrow \int d\left(\frac{1}{x^2}u\right) = \int \frac{2}{3}x^2 dx$$

$$\Rightarrow \frac{1}{x^2}u = \frac{2}{9}x^3 + C$$

$$\Rightarrow y^{\frac{2}{3}} = \frac{2}{9}x^5 + Cx^2 \Rightarrow y = \left(\frac{2}{9}x^5 + Cx^2\right)^{\frac{3}{2}}$$

5. 已知微分方程式為 $\frac{dy}{dx} = -(1+x+x^2) - (2x+1)y - y^2$

(1) 此為何種類型之微分方程式? (Clairaut、Bernoulli 或是 Riccati) (2%)

(2) 試求此微分方程式之解 $y(x) = ?$ (8%)

(1) $\frac{dy}{dx} = -(1+x+x^2) - (2x+1)y - y^2$ 此為 Riccati ODE

(2) 由觀察得一解 $S = -x$

令 $y = S + \frac{1}{V} = -x + \frac{1}{V} \Rightarrow \frac{dy}{dx} = -1 - \frac{V'}{V^2}$ 代回 ODE 可得

$$-1 - \frac{V'}{V^2} = -(1+x+x^2) - (2x+1)\left(-x + \frac{1}{V}\right) - \left(-x + \frac{1}{V}\right)^2$$

$\Rightarrow V' - V = 1 \longrightarrow$ 此為一階線性 ODE

可知其積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int dx} = e^{-x}$

同乘積分因子後可得 $e^{-x}V' - e^{-x}V = e^{-x} \Rightarrow \frac{d}{dx}(e^{-x}V) = e^{-x}$

$$\Rightarrow e^{-x}V = -e^{-x} + C$$

$$\Rightarrow V = -1 + Ce^x$$

$$\therefore y = S + \frac{1}{V} = -x + \frac{1}{Ce^x - 1}$$

6. 已知微分方程式為 $y = xy' + (y')^3$

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(2) 試求此微分方程式之解 $y(x) = ?$ (8%)

(1) $y = xy' + (y')^3$ 此為 Clairaut ODE

(2) $y = xy' + (y')^3$

令 $p = y' \Rightarrow y = xp + p^3$

將兩邊對 x 微分可得 $y' = p + xp' + 3p^2p'$

$$\Rightarrow p = p + xp' + 3p^2p'$$

$$\Rightarrow (x + 3p^2)p' = 0$$

由 $p' = 0 \Rightarrow p = c \Rightarrow y' = c$ 代回 $y = xy' + (y')^3$

可得 $y = xc + c^3$ (通解)

由 $x + 3p^2 = 0 \Rightarrow x = -3p^2 \Rightarrow x = -3(y')^2$ 代回 $y = xy' + (y')^3$

可得 $y = -2(y')^3 \Rightarrow (y')^3 = -\frac{1}{2}y \Rightarrow y' = \left(-\frac{1}{2}y\right)^{\frac{1}{3}}$ 代回 $y = xy' + (y')^3$

可得 $y = x\left(-\frac{1}{2}y\right)^{\frac{1}{3}} - \frac{1}{2}y \Rightarrow \frac{3}{2}y = x\left(-\frac{1}{2}y\right)^{\frac{1}{3}} \Rightarrow \left(\frac{3}{2}\right)^3 y^3 = -\frac{1}{2}x^3 y$

$$\Rightarrow 27y^2 = -4x^3 \text{ (特解)}$$

7. 試解下列各微分方程

(1) $y' + \frac{2x \sin y + y^3 e^x}{x^2 \cos y + 3y^2 e^x} = 0$ (8%)

(2) $x^2 (y')^2 + 4xyy' + 3y^2 = 0$ (8%)

(1) $y' + \frac{2x \sin y + y^3 e^x}{x^2 \cos y + 3y^2 e^x} = 0 \Rightarrow (2x \sin y + y^3 e^x)dx + (x^2 \cos y + 3y^2 e^x)dy = 0$

令 $M = 2x \sin y + y^3 e^x \Rightarrow \frac{\partial M}{\partial y} = 2x \cos y + 3y^2 e^x$

$N = x^2 \cos y + 3y^2 e^x \Rightarrow \frac{\partial N}{\partial x} = 2x \cos y + 3y^2 e^x$

由判斷式 $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ 可知，此為正合 ODE

\therefore 可知 $M = \frac{\partial \phi}{\partial x} = 2x \sin y + y^3 e^x \Rightarrow \phi = x^2 \sin y + y^3 e^x + f(y)$

$N = \frac{\partial \phi}{\partial y} = x^2 \cos y + 3y^2 e^x \Rightarrow \phi = x^2 \sin y + y^3 e^x + g(x)$

比較後可得 $\phi(x, y) = x^2 \sin y + y^3 e^x = C$

(2) $x^2 (y')^2 + 4xyy' + 3y^2 = 0 \Rightarrow (xy' + y)(xy' + 3y) = 0$

當 $xy' + y = 0 \Rightarrow y = \frac{C_1}{x} \Rightarrow y - \frac{C_1}{x} = 0$

當 $xy' + 3y = 0 \Rightarrow y = \frac{C_2}{x^3} \Rightarrow y - \frac{C_2}{x^3} = 0$

\therefore 此 ODE 之解為 $(y - \frac{C_1}{x})(y - \frac{C_2}{x^3}) = 0$