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Advanced Mathematics for Engineers

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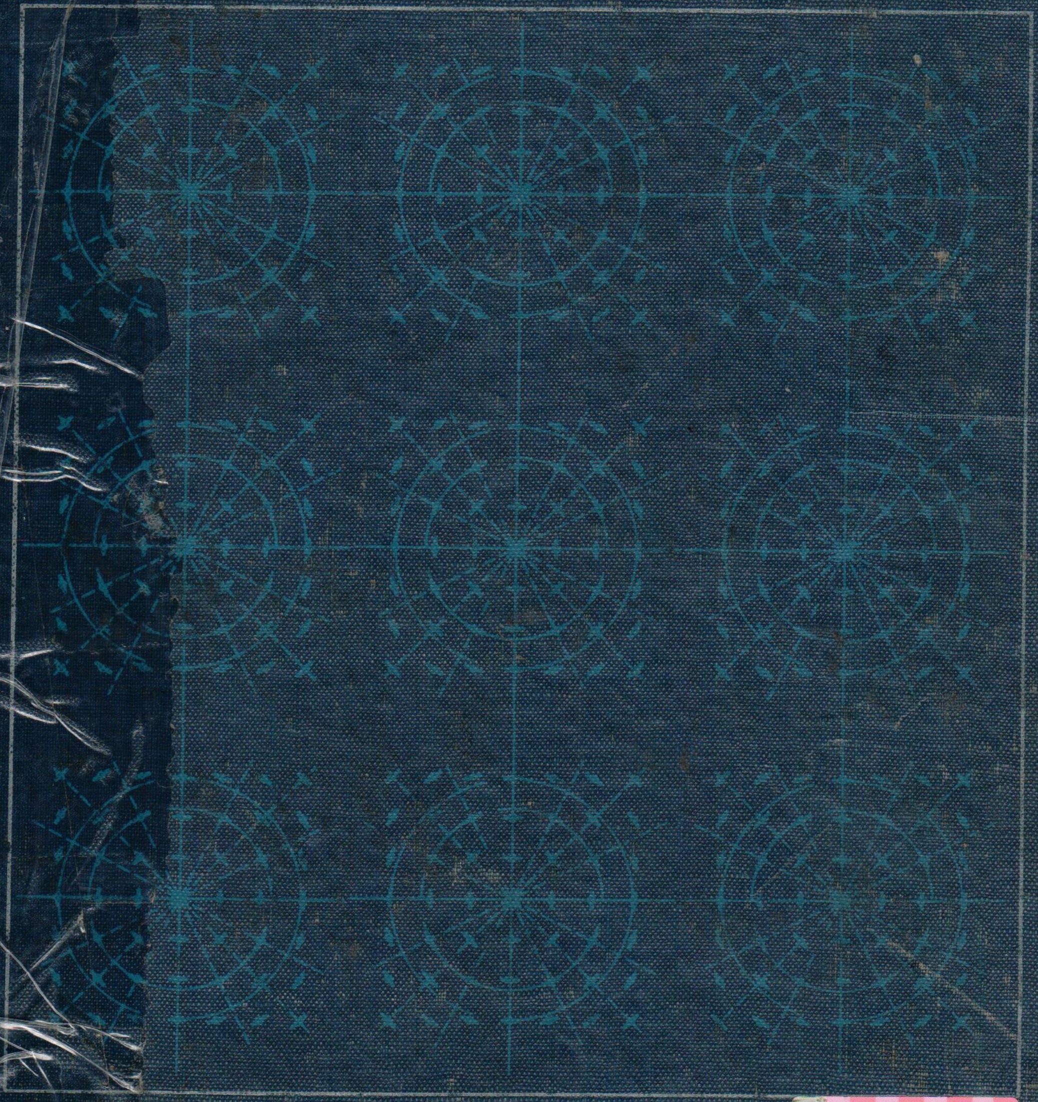

The solutions are thus given by the equation

$$y - \frac{y}{x^2 + y^2} = c.$$

We observe that for $c = 0$ the equation is satisfied when $y = 0$ and when $x^2 + y^2 = 1$. We can plot other solution curves by solving for x^2:

$$x^2 = 1 - y^2 + \frac{c}{y - c}$$

and selecting various values of c. The results are shown in Fig. 1–8. ◀

The condition (1–54) for exactness has wide physical application. If P and Q are interpreted as the x and y components of a vector (vector field in the xy-plane), then the condition $\partial P/\partial y = \partial Q/\partial x$ means that the vector field is *irrotational*. The conditions $\partial F/\partial x = P$, $\partial F/\partial y = Q$ state that the vector field is the *gradient* of a function. In physics such a function F (or its negative) is interpreted as the *potential* associated with the field. Thus our test for exactness is equivalent to the statement that every irrotational field in the plane is a gradient field or has a potential and, conversely, every gradient field is irrotational. The analogous statement for three-dimensional space is proved in Chapter 10.

$\frac{dy}{dx} = \frac{-3x - 2y}{x} =$

1–6 INTEGRATING FACTOR

If the equation $P dx + Q dy = 0$ is not exact, we can seek a function $\varphi(x, y)$ such that, after multiplication by $\varphi(x, y)$, the equation becomes exact. Such a function $\varphi(x, y)$ is called an *integrating factor.*

EXAMPLE 1 $(3x + 2y)\,dx + x\,dy = 0$. Here, after experimentation, we find that x is an integrating factor, for the equation

by $y' + \frac{2}{x} y = -3$

$$(3x^2 + 2xy)\,dx + x^2\,dy = 0$$

is exact ($\partial P/\partial y = \partial Q/\partial x = 2x$), and the general solution is seen by inspection to be

$$x^3 + x^2 y = c. \;◀$$

Finding integrating factors is often difficult, and considerable experience is needed. In fact, there is no guarantee that an integrating factor can be found (except as an infinite series or in some other form that is awkward to use).

Often it is helpful to group the terms and take advantage of known exact differentials such as those of xy, x/y, y/x, and

$$d\left[\tfrac{1}{2}\ln(x^2 + y^2)\right] = \frac{x\,dx + y\,dy}{x^2 + y^2}, \qquad d\tan^{-1}\frac{y}{x} = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

Also, an exact differential $P dx + Q dy = dF$ remains exact when multiplied by a function of F.

EXAMPLE 2 $[x(x^2 + y^2)^2 - y]\,dx + [(x^2 + y^2)^2 y + x]\,dy = 0$. As it stands, this is not an exact equation. We regroup and divide by $x^2 + y^2$:

$\frac{dy}{dx} = $ ______

$$(x^2+y^2)^2(xdx+ydy) - ydx + xdy = 0, \qquad \text{to } x^m y^n$$

divided by x^2+y^2

$$(x^2+y^2)(xdx+ydy) + \frac{-ydx+xdy}{x^2+y^2} = 0,$$

$$d\frac{(x^2+y^2)^2}{2} + d\tan^{-1}\frac{y}{x} = 0, \qquad \frac{(x^2+y^2)^2}{2} + \tan^{-1}\frac{y}{x} = c. \blacktriangleleft$$

or y'

EXAMPLE 3 $(3xy+2y^2)\,dx + (4x^2+5xy)\,dy = 0$. This is not exact and regrouping does not seem to help. We seek an integrating factor of form $x^m y^n$, with m and n to be found. After multiplication by such a factor, we have

$$P(x, y) = 3x^{m+1}y^{n+1} + 2x^m y^{n+2}, \qquad Q(x, y) = 4x^{m+2}y^n + 5x^{m+1}y^{n+1},$$

homogeneous case

$$\frac{\partial P}{\partial y} = 3(n+1)x^{m+1}y^n + 2(n+2)x^m y^{n+1},$$

$$\frac{\partial Q}{\partial x} = 4(m+2)x^{m+1}y^n + 5(m+1)x^m y^{n+1}.$$

Therefore we can make $\partial P/\partial y = \partial Q/\partial x$ if $3(n+1) = 4(m+2)$, $2(n+2) = 5(m+1)$. These are simultaneous equations for n and m: $3n - 4m = 5$, $2n - 5m = 1$, which are satisfied for $n = 3$, $m = 1$. Hence xy^3 is an integrating factor. Using it, we obtain

$$(3x^2y^4 + 2xy^5)\,dx + (4x^3y^3 + 5x^2y^4)\,dy = 0,$$

$$\frac{\partial F}{\partial x} = 3x^2y^4 + 2xy^5, \qquad F = x^3y^4 + x^2y^5 + g(y), \qquad \frac{\partial F}{\partial y} = 4x^3y^3 + 5x^2y^4,$$

so that we can take $g(y) = 0$. The solutions are given implicitly by

$$\boxed{x^3y^4 + x^2y^5 = c}. \blacktriangleleft$$

Problems (Section 1-6)

1. Verify that the equation is exact and find the general solution:
   a) $(x+2y)\,dx + (2x+3y)\,dy = 0$ b) $(5x-2y)\,dx + (7y-2x)\,dy = 0$
   c) $(\sin xy + xy\cos xy)\,dx + (x^2\cos xy + \sin y)\,dy = 0$
   d) $(3x^2 + y^2e^{xy})\,dx + (e^{xy} + xye^{xy})\,dy = 0$
   e) $\dfrac{xdx+ydy}{(x^2+y^2)^2} = 0$ f) $(xy)^3(ydx + xdy) = 0$
   g) $(x^4 + 6x^2y^2 + 2xy^3)\,dx + (4x^3y + 3x^2y^2 + y^4)\,dy = 0$
   h) $\dfrac{x^2-y^2}{x^2}dx + \dfrac{2y+2xy}{x}dy = 0$
2. Verify that the equation is exact and find the particular solution requested (if it exists):
   a) $(2x+y)\,dx + (x+2y)\,dy = 0$, $y = 1$ for $x = 1$
   b) $(4+2xy)\,dx + (x^2-4)\,dy = 0$, $y = 0$ for $x = 0$
   c) $(x^3+xy^2)\,dx + (x^2y+y^3)\,dy = 0$, $y = 0$ for $x = 0$
   d) $\dfrac{-ydx+xdy}{x^2+y^2} = 0$, $y = 0$ for $x = 0$
3. For a particle moving in the xy-plane subject to a force with x-component $P(x, y)$ and y-component $Q(x, y)$, the force is said to be derived from a potential $U(x, y)$ if