MATHEMATICAL THEORY OF WAVE MOTION

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ELLIS HORWOOD LIMITED

Publishers · Chichester

Halsted Press: a division of

JOHN WILEY & SONS

New York · Chichester · Brisbane · Toronto

2.4.3 Springy end

If the loop at $x = 0$ is attached to a spring which provides a transverse restoring force equal to Tk times the displacement from the equilibrium position (Fig. 2.4c), the boundary condition is $Tku(0, t) + Tu_x(0, t) = 0$.

Inserting the solution (2.13) we obtain

$$kf(-ct) + kg(ct) + f'(-ct) + g'(ct) = 0 \quad ,$$

whence

$$\frac{d}{d\zeta}\left\{ e^{k\zeta}g(\zeta) \right\} = e^{2k\zeta}\frac{d}{d\zeta}\left\{ e^{-k\zeta}f(-\zeta) \right\} \quad .$$

Integration by parts and substitution into (2.13) gives

$$u(x, t) = f(x - ct) + f(-x - ct) - 2ke^{-k(x+ct)}\int_0^{x+ct} e^{ks}f(-s)\mathrm{d}s \quad , \tag{2.16}$$

in which the lower limit of the integral has been chosen so as to satisfy (2.14).

2.4.4 Damped end

If the loop is attached to a damper which provides a transverse resisting force equal to R times the speed (Fig. 2.4d), the boundary condition is

$$Ru_t(0, t) + Tu_x(0, t) = 0 \quad .$$

We find that the reflected wave has the form

$$g(x + ct) = \frac{T - Rc}{T + Rc}f(-x - ct) \quad . \tag{2.17}$$

The interesting feature of (2.17) is that the reflected wave vanishes when $R = Z$, the characteristic impedance of the string defined by (2.11). When this occurs the impedance of the string is said to be matched by the impedance of the damper, and all the energy of the incident wave is dissipated in the damper. The matching of impedances is important in telegraphy where the impedances of electrical circuits must be matched in order to prevent a reflected pulse interfering with the signal which is being transmitted.

2.5 BOUNDARY BETWEEN TWO MEDIA

A different kind of boundary condition occurs when the medium in which a wave is travelling terminates at an interface with another medium in which the wave may also travel; for example, when an electromagnetic wave in air meets a dielectric or when sound waves in air meet an obstacle. This type of boundary condition is essentially different because whilst it produces a reflected wave in the first medium it also gives rise to a transmitted wave in the second medium.

2.5.1 Junction of two strings

Consider two semi-infinite strings S_1 and S_2, of linear densities ρ_1, ρ_2, joined at $x = 0$ and stretched at tension T, S_1 occupying the region $x < 0$ and S_2 the region $x > 0$. As the two strings have different linear densities it follows that they also have different wave speeds c_1 and c_2.

Let $f(x - c_1t)$ be a given incident wave in S_1, and let S_2 be initially undisturbed, so that $u(x, 0) = 0$ and $u_t(x, 0) = 0$ for $x > 0$. Then the wave in S_2, which may be written $h(x - c_2t) + H(x + c_2t)$, must satisfy $h'(\zeta) = 0$ and $H'(\zeta) = 0$ for $\zeta > 0$, and hence $H(x + c_2t)$ is a constant for $t > 0$. Therefore we may discard H and write

$$u(x, t) = \begin{cases} f(x - c_1t) + g(x + c_1t), & x < 0 \\ h(x - c_2t) & , x > 0 \end{cases} . \tag{2.18}$$

We call $h(x - c_2t)$ **the transmitted wave.**

We note that

$$f(\zeta) = h(\zeta) = 0 \ (\zeta > 0) \text{ and } g(\zeta) = 0 \ (\zeta < 0) \quad . \tag{2.19}$$

There are two boundary conditions at the junction $x = 0$. The first is the geometrical condition that the displacement must be continuous:

$$u(-0, t) = u(+0, t), \tag{2.20}$$

where $u(-0, t)$ denotes $\lim_{x \to 0} u(x, t)$ $(x < 0)$

and $u(+0, t)$ denotes $\lim_{x \to 0} u(x, t)$ $(x > 0)$.

The second is the dynamical condition that the transverse force must be continuous:

$$u_x(-0, t) = u_x(+0, t) \ ; \qquad Tu_x \ ? \tag{2.21}$$

this condition is necessary because a non-zero resultant force acting on the infinitesimally small mass at O would produce an infinite acceleration.

Inserting the solution (2.18) into (2.20) and (2.21) gives

$$f(-c_1t) + g(c_1t) = h(-c_2t) \tag{2.22}$$

and $$f'(-c_1t) + g'(c_1t) = h'(-c_2t) \quad .$$

Integrating we obtain

$$-\frac{1}{c_1} f(-c_1t) + \frac{1}{c_1} g(c_1t) = -\frac{1}{c_2} h(-c_2t) + 0 \quad , \tag{2.23}$$

the constant of integration being zero to satisfy (2.19).

Then, from (2.22) and (2.23),

$$g(\zeta) = \frac{c_2 - c_1}{c_2 + c_1} f(-\zeta) \tag{2.24}$$

and

$$h(\zeta) = \frac{2c_2}{c_2 + c_1} f\left(\frac{c_1}{c_2}\zeta\right) \quad . \tag{2.25}$$

Hence the reflected and transmitted waves are given by

$$g(x + c_1 t) = \frac{c_2 - c_1}{c_2 + c_1} f(-x - c_1 t) \quad ,$$

$$h(x - c_2 t) = \frac{2c_2}{c_2 + c_1} f\left(\frac{c_1}{c_2} x - c_1 t\right) \quad .$$

Note that, when $c_2 = 0$,

$$g(x + c_1 t) = -f(-x - c_1 t)$$

and $\quad h(x - c_2 t) = 0 \quad .$

So, by comparison with (2.15), this is equivalent to $x = 0$ being a fixed end to the string S_1. Similarly when c_2 is large the system is almost equivalent to $x = 0$ being a free end to the string S_1. In this case it is interesting to note that the amplitude of the transmitted wave cannot exceed twice that of the incident wave.

2.5.2 Energy transfer

An important feature of wave propagation is the transfer of energy through the supporting medium. We now consider what happens to the energy when a wave is reflected and transmitted at the junction. From (2.10) the energy flux in S_1 is

$$\begin{aligned} \mathcal{F} &= -Tu_x u_t = -T(f' + g')(-c_1 f' + c_1 g') \qquad (x < 0) \\ &= Tc_1 f'^2(x - c_1 t) - Tc_1 g'^2(x + c_1 t) \\ &= \mathcal{F}_\mathrm{I} - \mathcal{F}_\mathrm{R} \quad , \end{aligned}$$

where $\mathcal{F}_\mathrm{I}$ and $\mathcal{F}_\mathrm{R}$ may be termed the **incident flux** and the **reflected flux** respectively. The **transmitted flux** is $\mathcal{F}_\mathrm{T} = Tc_2 h'^2(x - c_2 t)$.