Cesaro sum for Fourier series

海大河海系 陳正宗

Fourier series for original function

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{p})+b_n sin(\frac{n\pi t}{p})\}$$

Fourier series for the derivative of original function

$$f'(t)=\sum_{n=1}^{\infty}\{a_n(\frac{-n\pi}{p})sin(\frac{n\pi t}{p})+b_n(\frac{n\pi}{p})cos(\frac{n\pi t}{p})\}=\sum_{n=1}^{\infty}\{a'_n cos(\frac{n\pi t}{p})+b'_n sin(\frac{n\pi t}{p})\}$$

Divergence for Fourier series representation of $f'(t)$ may occur when the termwise differetiation is not permissible.

Two methods can be employed:

(a). Alternative series by considering the jump value of the function. (Stokes' transformation)

(b). Cesaro sum treatment.

The general $C(k,r)$ Cesàro sum is defined as

$$S_k=C(k,r)\{\sum_{n=0}^{k}a_n\}\equiv\frac{C_{r-1}^{k+r-1}s_0+C_{r-1}^{k+r-2}s_1+\cdots+C_{r-1}^{r}s_{k-1}+C_{r-1}^{r-1}s_k}{C_r^{k+r}} \tag{1}$$

where $C_r^k=k!/(r!\,(k-r)!)$ and the partial sum is

$$s_k=\sum_{n=0}^{k}a_n \tag{2}$$

The $C(k,1)$ sum reduces to the conventional Cesàro sum:

$$S_k=C(k,1)\{\sum_{n=0}^{k}a_n\}\equiv\frac{s_0+s_1+\cdots+s_{k-1}+s_k}{k+1} \tag{3}$$

For the efficiency of computation, the s_i terms are changed to the a_i terms and the equation is thus changed to

$$S_k=C(k,1)\{\sum_{n=0}^{k}a_n\}\equiv\frac{1}{k+1}\sum_{n=0}^{k}(k-n+1)\ a_n \tag{4}$$

Similarly, the $C(k,2)$ Cesàro sum is

$$S_k=C(k,2)\{\sum_{n=0}^{k}a_n\}\equiv\frac{1}{(k+1)(k+2)}\sum_{n=0}^{k}(k-n+1)(k-n+2)\ a_n \tag{5}$$

In the same way, the $C(k,3)$ and $C(k,4)$ Cesàro sums are respectively

$$S_k=C(k,3)\{\sum_{n=0}^{k}a_n\}\equiv\frac{\sum_{n=0}^{k}(k-n+1)(k-n+2)(k-n+3)\ a_n}{(k+1)(k+2)(k+3)}$$

$$S_k = C(k,4)\{\sum_{n=0}^{k} a_n\} \equiv \frac{\sum_{n=0}^{k}(k-n+1)(k-n+2)(k-n+3)(k-n+4)\ a_n}{(k+1)(k+2)(k+3)(k+4)}$$

If the a_0 term is missing, the $C(k,1)$ Cesàro sum reduces to

$$S_k \equiv C(k,1)\{\sum_{n=1}^{k} a_n\} = \frac{1}{k}\sum_{n=1}^{k}(k-n+1)\ a_n \tag{6}$$