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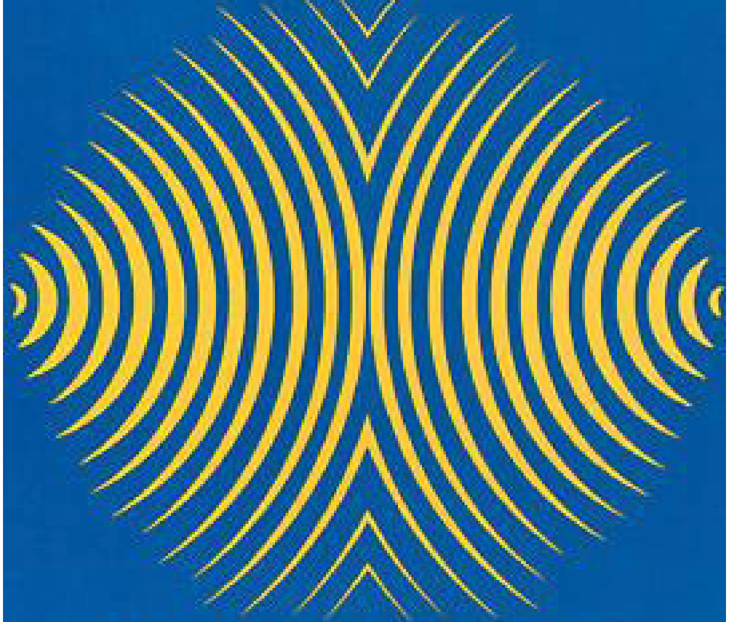

PHYSICS OF WAVES

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where $E_1 = K_1 + V_1$ is the total energy density and $\nabla \cdot \mathbf{P}$ is the divergence of the directed power flow per unit area. Show that the expressions for K_1, V_1, and P, as given by (1.8.1), (1.8.4), and (1.8.13), satisfy (1.8.32).

1.9 The Reflection and Transmission of Waves at a Discontinuity

We are here concerned with transverse waves on a string consisting of two parts, as shown in Fig. 1.9.1. The left part has a linear mass density λ_1 and the right part a different linear mass density λ_2, with both parts under the same tension τ_0. For convenience we place the x origin at the discontinuity. We suppose that a source of sinusoidal waves on the negative x axis is sending waves toward the discontinuity and that the waves continuing past it are absorbed with no reflection by a distant sink. We wish to examine how the abrupt change in properties of the string affects the passage of waves down the string.

Our first task is to find the so-called *boundary conditions* that the wave motion must satisfy at the discontinuity. Evidently there must exist two independent conditions, reflecting the fact that the differential wave equation is of second order. One of these is obviously the continuity of the string, i.e., of its displacement, or, equivalently, the continuity of its transverse velocity. The other is the continuity of the transverse force in the string, as given by (1.8.12). This boundary condition is basically a consequence of Newton's third law. If the force is not continuous at the boundary, an infinitesimal mass there would be subject to a finite force, resulting in an infinite acceleration. Accordingly for all times at $x = 0$, we require that

$$\eta_{\text{left}} = \eta_{\text{right}}$$

$$\left(-\tau_0 \frac{\partial \eta}{\partial x}\right)_{\text{left}} = \left(-\tau_0 \frac{\partial \eta}{\partial x}\right)_{\text{right}}. \tag{1.9.1}$$

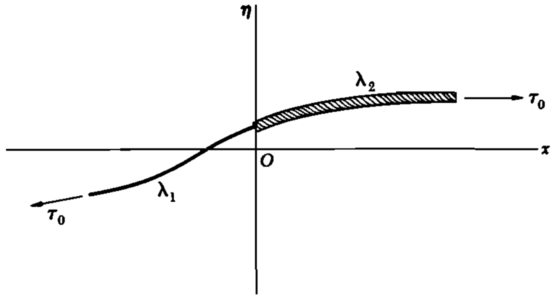

Fig. 1.9.1 Wave on string having a discontinuity in mass density at the origin.

Let us take the incident wave coming from the left to be the real part of

$$\eta_1 = A_1 e^{i(\kappa_1 x - \omega t)} \qquad -\infty < x < 0, \tag{1.9.2}$$

which has a specified amplitude A_1 and the velocity $c_1 = \omega/\kappa_1 = (\tau_0/\lambda_1)^{1/2}$. The wave transmitted past the discontinuity is assumed to be the real part of

$$\eta_2 = \check{A}_2 e^{i(\kappa_2 x - \omega t)} \qquad 0 < x < \infty, \tag{1.9.3}$$

which has a complex amplitude $\check{A}_2$ yet to be determined and a velocity and wave number that differ from those of the first wave, $c_2 = \omega/\kappa_2 = (\tau_0/\lambda_2)^{1/2}$. Both waves must necessarily have the same frequency.

We now discover that it is impossible with only these two waves to satisfy the boundary conditions (1.9.1), since the first condition would require that $A_1 = \check{A}_2$ and the second that $\kappa_1 A_1 = \kappa_2 \check{A}_2$. Necessarily, then, there must exist a *third* wave that is reflected from the boundary, in order that the boundary conditions (1.9.1) be satisfied.

We assume that the reflected wave traveling to the left is the real part of

$$\eta_1' = \check{B}_1 e^{i(-\kappa_1 x - \omega t)} \qquad -\infty < x < 0, \tag{1.9.4}$$

where $\check{B}_1$ is to be determined and the wave number is that appropriate to the string on the left side of the boundary.

The boundary conditions now require that

$$\begin{aligned} A_1 + \check{B}_1 &= \check{A}_2 \\ \kappa_1 A_1 - \kappa_1 \check{B}_1 &= \kappa_2 \check{A}_2, \end{aligned} \tag{1.9.5}$$

which are sufficient to determine $\check{B}_1$ and $\check{A}_2$ in terms of A_1, the amplitude of the incident wave. Solving for the amplitude ratios $\check{B}_1/A_1$ and $\check{A}_2/A_1$, which are defined as the complex *amplitude reflection coefficient* $\check{R}_a$ and *amplitude transmission coefficient* $\check{T}_a$, respectively, we find that

$$\begin{aligned} \check{R}_a &\equiv \frac{\check{B}_1}{A_1} = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \\ \check{T}_a &\equiv \frac{\check{A}_2}{A_1} = \frac{2\kappa_1}{\kappa_1 + \kappa_2} = \frac{2Z_1}{Z_1 + Z_2}, \end{aligned} \tag{1.9.6}$$

where we have expressed the results in terms of the characteristic impedances $Z_1\,[=\lambda_1 c_1 = (\lambda_1 \tau_0)^{1/2}]$ and Z_2 of the two parts of the string. The fact that $\check{R}_a$ and $\check{T}_a$ turn out to be real indicates that the reflected and transmitted waves are not shifted in phase, except for a possible 180° phase shift for the reflected wave, when $\check{R}_a$ is negative. We note that if $\lambda_1 > \lambda_2$, $\check{R}_a$ is positive, which implies that the reflected wave has the same phase as the incident wave, whereas if $\lambda_1 < \lambda_2$, $\check{R}_a$ is negative, showing that the reflected and incident waves are 180° out of phase. Since $\breve{T}_a$ is always positive, the transmitted wave has the same phase as the incident wave.

It is also customary to define a *power reflection coefficient* R_p and a *power transmission coefficient* T_p to express the reflection and transmission of waves at a boundary. The power carried by a traveling sinusoidal wave is given by (1.8.11). Hence for the power reflection coefficient

$$R_p = \frac{\frac{1}{2}\lambda_1 c_1 \omega^2 B_1^2}{\frac{1}{2}\lambda_1 c_1 \omega^2 A_1^2} = \frac{B_1^2}{A_1^2} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2, \tag{1.9.7}$$

and for the power transmission coefficient

$$T_p = \frac{\frac{1}{2}\lambda_2 c_2 \omega^2 A_2^2}{\frac{1}{2}\lambda_1 c_1 \omega^2 A_1^2} = \frac{\lambda_2 c_2 A_2^2}{\lambda_1 c_1 A_1^2} = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2}. \tag{1.9.8}$$

The fact that the incident power equals the reflected power plus the transmitted power is expressed by $R_p + T_p = 1$. Since (1.9.6) to (1.9.8) depend only on properties of the medium (the string) and not on the frequency of the waves, they must hold for waves of arbitrary shape. Reflection and transmission coefficients for *plane* waves of any sort incident normally on a plane boundary between two media have the same form as those found here when expressed in terms of the characteristic impedances of the media.

Problems

1.9.1 Obtain the boundary conditions (1.9.5) from (1.9.1) and show that they lead to (1.9.6).

1.9.2 A uniform string of linear mass density λ_0 and under a tension τ_0 has a small bead of mass m attached to it at $x = 0$. Find expressions for the complex amplitude and the power reflection and transmission coefficients for sinusoidal waves brought about by the mass discontinuity at the origin. Do these coefficients hold for a wave of arbitrary shape?

1.9.3 Three long identical strings of linear mass density λ_0 are joined together at a common point forming a symmetrical Y. Thus they lie in a plane 120° apart. Each is given the same tension τ_0. A distant source of sinusoidal waves sends transverse waves, with motion perpendicular to the plane of the strings, down one of the strings. Find the reflection and transmission coefficients that characterize the junction.

1.9.4 A long string under tension τ_0 having a linear mass density λ_1 is tied to a second string with linear mass density $\lambda_2 \ll \lambda_1$. Transverse waves on the heavy string are incident on the junction. Find what happens to them.