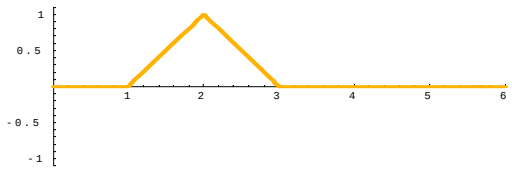
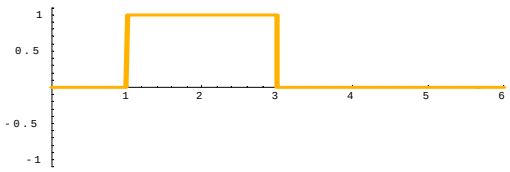
海洋大學河海工程學系 2005 工程數學(四)有限固定繩波

Fixed string: $c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$, B.C. (Boundary Condition): $u(0,t)=0,\ u(l,t)=0$

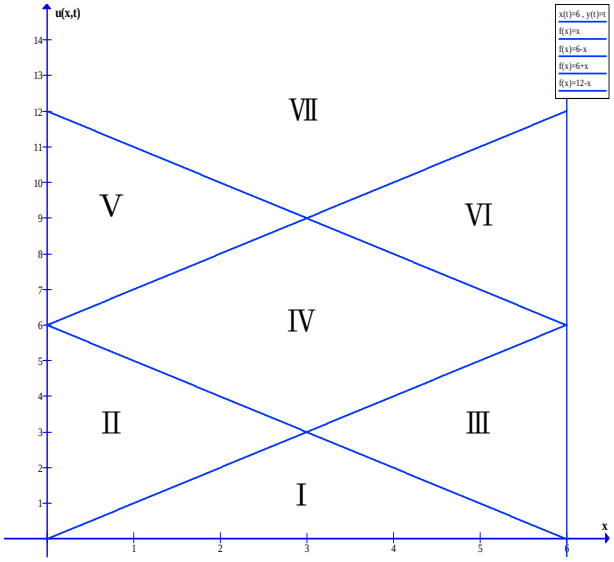
I.C. (Initial Condition): $u(x,0)=\phi(x),\ u_t=\varphi(x)=0$

Triangle Wave: $\phi(x)=\begin{cases}x-1, & 1\le x<2\\ 3-x, & 2\le x\le 3\\ 0, & otherwise\end{cases}$ ； Square Wave: $\phi(x)=\begin{cases}1, & 1\le x\le 3\\ 0, & otherwise\end{cases}$





1. Diamond Rule： $f(x)=\phi(x)+\varphi(x)$



$$u_1(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]$$

$$u_2(x,t)=\frac{1}{2}[f(x+ct)+f(-x+ct)]$$

$$u_3(x,t)=\frac{1}{2}[f(x-ct)-f(2l-(x+ct))]$$

$$u_4(x,t)=-\frac{1}{2}[f(-x+ct)+f(2l-(x+ct))]$$

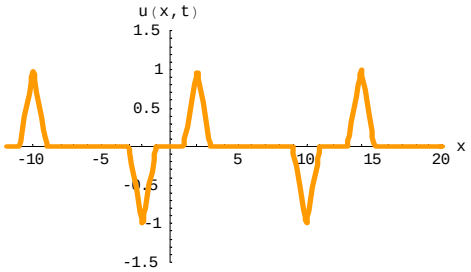
$$u_5(x,t)=-\frac{1}{2}[f(2l-(x+ct))-f(2l-(-x+ct))]$$

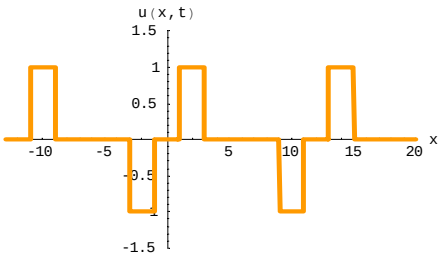
$$u_6(x,t)=-\frac{1}{2}[f(-x+ct)-f(-2l+(x+ct))]$$

$$u_7(x,t)=\frac{1}{2}[f(2l-(-x+ct))+f(-2l+(x+ct))]$$

2. Image Method：

以 l 爲一單位長，每一單位取一區間，而每一區間產生一個反對稱的影像。依據 Huygen's Principle 可知，其所產生之反對稱影像即爲新的波源，再利用新波源去產生反對稱影像，以此原則不斷往外產生影像，即可得下列圖形。





3. 一杆進洞法 (Fourier Method)：

$$u(x,t)=X(x)T(t)=\sum_{n=1}^{\infty}\sin(\frac{n\pi x}{l})\times P_n\cos(\frac{n\pi t}{l})$$

$$P_n=\frac{1}{l}\int_0^l\phi(x)\sin(\frac{n\pi x}{l})dx \quad\Rightarrow\quad \text{Triangle Wave: } P_n=\frac{-12}{(n\pi)^2}[\sin(\frac{n\pi}{2})-2\sin(\frac{n\pi}{3})+\sin(\frac{n\pi}{6})]$$

$$\text{Square Wave: } P_n=\frac{-2}{n\pi}[\cos(\frac{n\pi}{2})-\cos(\frac{n\pi}{6})]$$

Fixed string by Mathematica:

1. Diamond Rule：

Triangle Wave:

```
f[x_] := x - 1 /; 1 ≤ x < 2
f[x_] := 3 - x /; 2 ≤ x ≤ 3
f[x_] := 0 /; (x > 3 || x < 1)
```

Square Wave:

```
f[x_] := If[1 ≤ x ≤ 3, 1, 0]
```

```
u1[x_] := If[(0 ≤ x ≤ 6 && x - t ≥ 0 && x + t ≤ 6), 1/2 (f[x + t] + f[x - t]), 0]
u2[x_] := If[(0 ≤ x && x - t <= 0 && x + t <= 6), 1/2 (f[x + t] - f[-x + t]), 0]
u3[x_] := If[(x ≤ 6 && x - t ≥ 0 && x + t >= 6), 1/2 (f[x - t] - f[12 - (x + t)]), 0]
u4[x_] := If[(x - t <= 0 && x + t ≥ 6 && x + t <= 12 && x - t ≥ -6), -1/2 (f[12 - (x + t)] + f[-x + t]), 0]
u5[x_] := If[(0 ≤ x && x - t ≤ -6 && x + t ≤ 12), -1/2 (f[12 - (x + t)] - f[12 - (-x + t)]), 0]
u6[x_] := If[(x ≤ 6 && x - t ≥ -6 && x + t ≥ 12), -1/2 (f[-x + t] - f[-12 + (x + t)]), 0]
u7[x_] := If[(x - t ≤ -6 && x + t ≥ 12 && x + t <= 18 && x - t ≥ -12), 1/2 (f[12 - (-x + t)] + f[-12 + (x + t)]), 0]
```

2. Image Method：

Triangle Wave:

```
s[x_] := If[1 ≤ x < 2, x - 1, If[2 ≤ x ≤ 3, 3 - x, 0]]
im1[x_] := If[-3 ≤ x < -2, -3 - x, If[-2 ≤ x ≤ -1, x + 1, 0]]
im2[x_] := If[9 ≤ x < 10, 9 - x, If[10 ≤ x ≤ 11, x - 11, 0]]
im3[x_] := If[-11 ≤ x < -10, 11 + x, If[-10 ≤ x ≤ -9, -9 - x, 0]]
im4[x_] := If[13 ≤ x < 14, x - 13, If[14 ≤ x ≤ 15, 15 - x, 0]]
```

Square Wave:

```
s[x_] := If[1 ≤ x ≤ 3, 1, 0]
im1[x_] := If[-3 ≤ x ≤ -1, -1, 0]
im2[x_] := If[9 ≤ x ≤ 11, -1, 0]
im3[x_] := If[-11 ≤ x ≤ -9, 1, 0]
```

3. 一杆進洞法 (Fourier Method)：

Triangle Wave:

$$u[x\_] := \sum_{n=1}^{k} \left(\text{Sin}\left[\frac{n*\pi*x}{6}\right] * \left(\frac{12}{(n*\pi)^2} * \left(-\text{Sin}\left[\frac{n*\pi}{2}\right] + 2*\text{Sin}\left[\frac{\pi*n}{3}\right] - \text{Sin}\left[\frac{\pi*n}{6}\right] \right) * \text{Cos}\left[\frac{n*\pi*t}{6}\right] \right) \right)$$

Square Wave:

$$u[x\_] := \sum_{n=1}^{k} \left(\text{Sin}\left[\frac{1}{6}(n*\pi*x)\right] * \left(-2/(n*\pi) * \left(\text{Cos}\left[\frac{1}{2}(n*\pi)\right] - \text{Cos}\left[\frac{1}{6}(\pi*n)\right] \right) * \text{Cos}\left[\frac{1}{6}(n*\pi*t)\right] \right) \right)$$