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STUDY ON THE SURFACE FAST MULTIPOLE BOUNDARY ELEMENT METHOD BASED ON SPHERICAL HARMONIC SPACE — MATHEMATICAL THEORY PART —

Chunxiao Yu
College of Science
Yanshan University
Qinhuangdao 066004, P. R. China
chxy@ysu.edu.cn

Guangxian Shen
College of Mechanical Engineering
Yanshan University
Qinhuangdao 066004, P. R. China
sgx35@ysu.edu.cn

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Abstract. *The Fast Multipole Method (FMM) is applied for the boundary surface of 3-D elasticity for the first time. On the mathematical basis of current FMM and Boundary Element Method (BEM), a spherical harmonic function and some related Fast Multipole Boundary Element Method (FM-BEM) numerical formulas are constructed for the boundary surface. Fundamental theorems of the FM-BEM are presented and proved. Then a complete FMM-BEM theoretical system for 3-D elasticity is preliminarily established, which provides strong mathematical support for further promotion of the FM-BEM in rolling engineering field and other fields.*
Keywords: FMM, BEM, FM-BEM, Spherical harmonic function

1. **Introduction.** The so called Fast Multipole Method (FMM)[1] proposed by L. Greengard originates from a quantitative description of a multi-body electrostatic field. It is suitable for the rapid computation of interacting potentials among a large number of particle sets. Recently, the FMM has been applied in mathematics, mechanics and in other fields because of the small computation and memory requirement. On the international scene, the Fast Multipole Boundary Element Method (FM-BEM) [2-4] is the latest development mainly using the FMM to rapidly compute the summation term $\sum_{i=1}^{n} \frac{1}{R_i}$in discrete boundary integral equations and to enlarge the solution scale.

For the study of the FM-BEM, Z. H. Yao's groups [5-7] in Tsinghua University have applied the FMM for 2-D and 3-D elasticity Boundary Element Method (BEM) using Taylor serials expansion and successfully simulated the deformation—stress field of different particles for composite materials. G. X. Shen's groups [8-11] have applied the FMM for 3-D elasticity, 3-D elasto-plasticity and 3-D contact BEM problems, completing the large-scale simulation of rolling problems. J. T. Chen and K. H. Chen [12] have applied the FMM for Double BEM (DBEM) to accelerate the construction of influence matrix and to solve 2-D large-scale acoustic problems. However, these studies have not involved the application of the FMM for rapid computation of 3-D boundary surface BEM problems.

Therefore, the FMM suitable for discrete boundary integral domain terms is utilized for 3-D boundary surface in this work. The motivation for this research is to construct and prove some related numerical formulas and fundamental theorems, to develop the BEM and finally establish the FM-BEM theory, and to analyze the superiority of the FM-BEM. The newly proposed FM-BEM will be a numerical computational method used for rolling engineering field and other fields with high efficiency. Its application prospect is extremely extensive.

2. Fundamental FM-BEM Formulas for 3-D Elasticity.

2.1. BEM fundamental solution. The partial differential equation for 3-D elasticity can be transformed into boundary integral equation using Kelvin's fundamental solution and Betti's reciprocity theorem. The boundary integral equation for 3-D elasticity without body force is written as [3]

$$c_{ij}\, u_j(x) + \int_\Gamma T_{ij}(x,y)\, u_j(y)\, d\Gamma = \int_\Gamma U_{ij}(x,y)\, t_j(y)\, d\Gamma \tag{1}$$

where Γ denotes the boundary, x and y stand for the source point and observation point, respectively. $x,\ y \in \Gamma$. The indices $i,\ j = 1,\ 2,\ 3$. c_{ij} are related to the geometry information at the source point x. u_j and t_j specify boundary displacement and traction, respectively. $U_{ij}(x,y)$ and $T_{ij}(x,y)$ are kernel functions for 3-D elasticity, they are expressed as:

$$U_{ij}(x,y) = \frac{1}{16\pi(1-\nu)\mu R}[(3-4\nu)\delta_{ij} + R_{,i}\, R_{,j}] \tag{2}$$

$$T_{ij}(x,y) = \frac{-1}{8\pi(1-\nu)R^2}\left\{ \frac{\partial R}{\partial n}[(1-2\nu)\,\delta_{ij} + 3\, R_{,i}\, R_{,j}] - (1-2\nu)(n_j\, R_{,i} - n_i\, R_{,j}) \right\} \tag{3}$$

where δ_{ij} is the Kronecker function, R is the distance between y and x, $R = y - x$. μ is the shear modulus and ν is the Poisson's ratio. n is the outward normal vector of the boundary.

At the source point x^q, the numerical integration for Eq. (1) is expressed as:

$$c_{ij}\, u_j(x^q) + \sum_{k,l,s} T_{ij}[x^q, y(\xi^s)]\, u_j^{kl} \phi^l(\xi^s) J[y(\xi^s)]\, \omega^s - \sum_{k,l,s} U_{ij}[x^q, y(\xi^s)]\, t_j^{kl} \phi^l(\xi^s) J[y(\xi^s)]\, \omega^s = 0 \tag{4}$$

where ξ denotes the local coordinate, q and s stand for the source point and observation point, respectively. u_j^{kl} and t_j^{kl} specify the displacement and traction of the node l of element k, respectively. $\phi^l(\xi)$ denotes the shape function of node l, ω^s denotes the weight coefficient at ξ^s.

2.2. Discretization and solution of FM-BEM equations. From Eq. (4), we can see that the operation $\sum_{l,s} \phi^l(\xi^s) J[y(\xi^s)]\, \omega^s$ for each element domain is invariable during the process of all integrations. If iterative method is selected to solve the equation, the two summation terms $\sum_{k,l,s} u_j^{kl} \phi^l(\xi^s) J[y(\xi^s)]\, \omega^s$ and $\sum_{k,l,s} t_j^{kl} \phi^l(\xi^s) J[y(\xi^s)]\, \omega^s$ in Eq. (4) are constants in each iteration for every element because u_j^{kl} and t_j^{kl} are assigned before iteration. For Eq. (4), the structures of two summation terms are the same. So only one is required to discuss. For the first summation term in Eq. (4), let

$$C_k^s = \sum_{k,l,s} u_j^{kl} \phi^l(\xi^s) J[y(\xi^s)]\, \omega^s$$

If the fundamental solution $T_{ij}[x^q, y(\xi^s)]$ is expressed as the function of $1/R^s$, namely,

$$T_{ij}[x^q, y(\xi^s)] = f_{ij}(x^q)(1/R^s)$$

the first summation term in Eq. (4) can be written as

$$\sum_{k,l,s} T_{ij}[x^q, y(\xi^s)]\, u_j^{kl} \phi^l(\xi^s)\, J[y(\xi^s)]\, \omega^s = \sum_{k,s} f_{ij}(x^q)(1/R^s) C_k^s$$

$$= \sum_{k,s} f_{ij}(x^q)(C_k^s/R^s) = f_{ij}(x^q) \sum_{k,s} C_k^s/R^s \tag{5}$$

where $R^s = \|x^q - y(\xi^s)\|$. It is obvious that the summation term in Eq. (5) can be computed using the FMM.

Eq. (4) is rewritten as

$$\begin{aligned} & c_{ij}\, u_j(x^q) + R_{ijp}(x^q) \sum_{k,l,s} \frac{1}{|\, x^q - y(\xi^s)\,|} \left\{ u_j^{kl}\, \phi^l(\xi^s)\, n_p[y(\xi^s)]\, J[y(\xi^s)]\, \omega^s \right\} \\ & + S_{ij}(x^q) \sum_{k,l,s} \frac{1}{|\, x^q - y(\xi^s)\,|} \left\{ u_j^{kl}\, \phi^l(\xi^s)\, n_p[y(\xi^s)]\, y_j\, J[y(\xi^s)]\, \omega^s \right\} \\ & - P_{ij}(x^q) \sum_{k,l,s} \frac{1}{|\, x^q - y(\xi^s)\,|} \left\{ t_j^{kl}\, \phi^l(\xi^s)\, J[y(\xi^s)]\, \omega^s \right\} \\ & - Q_i(x^q) \sum_{k,l,s} \frac{1}{|\, x^q - y(\xi^s)\,|} \left\{ t_j^{kl}\, \phi^l(\xi^s)\, y_j(\xi^s)\, J[y(\xi^s)]\, \omega^s \right\} = 0 \end{aligned} \tag{6}$$

In Eq. (6), $j = 1\,,\ 2\,,\ 3$, $p = 1\,,\ 2\,,\ 3$. So the FMM is required to call 9 times for the first summation term, 3 times for the second term, 3 times for the third term and one time for the fourth term. It totals 16 times.

The boundary $\Gamma$is discretized into NUM_E boundary elements and NUM_N boundary nodes. When the boundary conditions are introduced, the equation system with 3×NUM_N unknowns is established as

$$Ax = b \tag{7}$$

Eq.(7) is solved by the combination of the FMM and Generalized Minimum Residual Method (GMRES) [10].

3. **FM-BEM Spherical Harmonic Function and Numerical Computation.**

3.1. **Spherical harmonic function and addition formula.** This part gives some important formulas to be used in the following parts.

Definition 3.1 [13]. *In polar coordinate system, Laplace equation*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=0 \tag{8}$$

is solved using Separation of Variables. Let $V(r,\theta,\varphi)=R(r)\,S(\theta,\varphi)$*, we have*

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{\lambda}{r^2}R=0 \tag{9}$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial S}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2}+\lambda S=0 \tag{10}$$

where λ *is a constant. Eq. (10) has bounded periodic solutions with* $0\le\theta\le\pi$ *and* $0\le\varphi\le 2\pi$ *if and only if* $\lambda=n(n+1)$, $n=0,1,2,\cdots$. *Such solutions are expressed as* $S_n(\theta,\varphi)$ *and called n-time spherical harmonic functions.*

Theorem 3.1. *When the direction of polar axis is changed in spherical polar coordinate system, we have the following addition formula*

$$P_n(\cos\gamma)=\sum_{m=-n}^{n}(-1)^m P_n^m(\cos\theta)\,P_n^{-m}(\cos\theta')\,e^{im(\varphi-\varphi')} \tag{11}$$

$$=\sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\,P_n^m(\cos\theta')\,e^{im(\varphi-\varphi')} \tag{12}$$

$$=P_n(\cos\theta)\,P_n(\cos\theta')+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\,P_n^m(\cos\theta')\,\cos m(\varphi-\varphi') \tag{13}$$

where

$$\cos\gamma=\cos\theta\,\cos\theta'+\sin\theta\,\sin\theta'\,\cos(\varphi-\varphi') \tag{14}$$

γ *is the included angle of* OP *(directed by* θ,φ*) and* OP'*(directed by* θ',φ'*).* $P_n^m(x)$ *is m-order n-time associated Legendre function of the first kind, which is defined by Hobson.*

$$P_n^m(x)=(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)\qquad(0\le m\le n,\ -1\le x\le 1) \tag{15}$$

where $(1-x^2)^{\frac{m}{2}}$ *is positive. Using Rodrigues formula, the associated Legendre function* $P_n^m(x)$ *can be proved to satisfy the relationship*

$$P_n^{-m}(x)=(-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(x) \tag{16}$$

and the orthogonal relation$(m,\ m'\ge 0)$

$$\int_{-1}^{1}P_n^m(x)\,P_{n'}^m(x)\,dx=\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'} \tag{17}$$

3.2. **Selection of the FM-BEM spherical harmonic function.** From **Theorem 3.1**, we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial S}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2}+n(n+1)S=0 \tag{18}$$

or

$$\frac{\partial^2 S}{\partial\theta^2}+\cot\theta\frac{\partial S}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2}+n(n+1)S=0 \qquad (18)*$$

The bounded (for θ) periodic (for φ) solutions totals $2n+1$ with $0\le\varphi\le 2\pi$ and $0\le\theta\le\pi$.

$$P_n^m(\cos\theta)e^{im\varphi},\quad m=0,\ \pm1,\ \cdots,\ \pm n$$

In the FM-BEM, the spherical harmonic function

$$Y_{nm}(\theta,\ \varphi)=\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\theta)e^{im\varphi}\quad(m=0,\ \pm1,\ \cdots,\ \pm n) \tag{19}$$

is taken as the bounded periodic solution of Eq. (18), which satisfies the relationship

$$\int_0^\pi\int_0^{2\pi} Y_{nm}^* Y_{n'm'}\sin\theta\,d\varphi\,d\theta=\delta_{mm'}\delta_{nn'} \tag{20}$$

where Y_{nm}^* is the conjugate complex of Y_{nm}, and

$$Y_{nm}^*=(-1)^m Y_{n,\,-m} \tag{21}$$

Eq. (21) can be proved by Eq. (16). Eq. (20) can be proved by Eq. (17) and the following relationship.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-imx}e^{inx}dx=\delta_{mn}\quad(\rho(x)=1) \tag{22}$$

However, Eq. (16) is required if $m<0$ or $m'<0$.

3.3. **Numerical computation of the FM-BEM spherical harmonic function.** In the FM-BEM, Eq. (19) is the bond of spherical harmonic function and associated Legendre function. For the numerical computation of the spherical harmonic function $Y_{nm}(\theta,\ \varphi)$ in Eq. (19), the key point is the associated Legendre polynomial. There are many numerical methods to compute the associated Legendre polynomial. However, most of them do not work too well. For example, there is an explicit expression

$$P_n^m(x)=\frac{(-1)^m(n+m)!}{2^m m!(n-m)!}(1-x^2)^{\frac{m}{2}}[1-\frac{(n-m)(m+n+1)}{1!(m+1)}(\frac{1-x}{2}) + \frac{(n-m)(n-m-1)(m+n+1)(m+n+2)}{2!(m+1)(m+2)}(\frac{1-x}{2})^2-\cdots\cdots] \tag{23}$$

where only the terms including $(1-x)^{n-m}$ are used. Because the sign alternates term by term, continuous terms with contrary sign can be cancelled out in the solution. When n is very big, the isolated term will be much bigger than the summation with poor precision. It is an unsatisfactory method. Therefore, a more effective computational method is needed, which is given as follows.

The associated Legendre polynomial satisfies many recurrence relationships, which can be with respect to n or m, or to both n and m. Most of them with respect to m are instable and not suitable for numerical computation. Using mathematical induction, $P_n^m(x)$ is proved to satisfy

$$(n-m)\,P_n^m(x) = (2n-1)\,x\,P_{n-1}^m(x) - (n+m-1)\,P_{n-2}^m(x) \tag{24}$$

which is stable with respect to n and also a practical numerical formula. It can be proved that $P_n^m(x)$ has closed expression

$$P_m^m(x) = (-1)^m(2m-1)!!\,(1-x^2)^{\frac{m}{2}} \tag{25}$$

with respect to initial values. For Eq. (24), let $n = m+1$, $P_{m-1}^m(x) = 0$, then

$$P_{m+1}^m(x) = (2m+1)\,x\,P_m^m(x) \tag{26}$$

In Eq. (24), the two initial values suitable for general n are given by Eqs. (25) and (26).

4. FM-BEM Theorems.

4.1. Fundamental theorems for the surface FM-BEM. All of the surface FM-BEM formulas are established in a spherical polar coordinate system, as shown in Figure 1. To apply the FMM for the boundary, define m-order n-time spherical harmonic function $Y_{nm}(\theta,\ \varphi)$ according to Eq. (19). Fundamental theorems for the surface FM-BEM are presented and proved as follows.

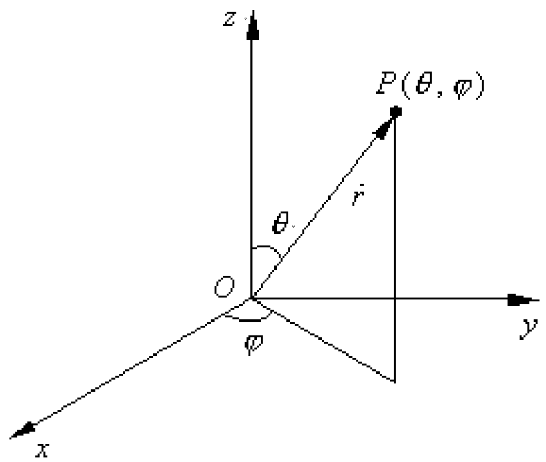


FIGURE 1. Spherical polar coordinate system.

Theorem 4.1. *Assume that there are N charges $X_1,\ X_2,\ \cdots,\ X_N$ with intensity $q_1,\ q_2,\ \cdots,\ q_N$ on surface S, whose spherical polar coordinates are $(r_1', \theta_1', \varphi_1'), (r_2', \theta_2', \varphi_2'), \cdots, (r_N', \theta_N', \varphi_N')$, respectively, as shown in Figure 2. For a random point $X(r,\ \theta,\ \varphi) \in R^3$ satisfying $r > \max\limits_{1\le i\le N} \{r_i'\}$, the potential $\Phi(X)$ generated from charges $q_1,\ q_2,\ \cdots,\ q_N$ is*

$$\Phi(X) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{M_n^m}{r^{n+1}}\, Y_{nm}(\theta,\ \varphi) \tag{27}$$

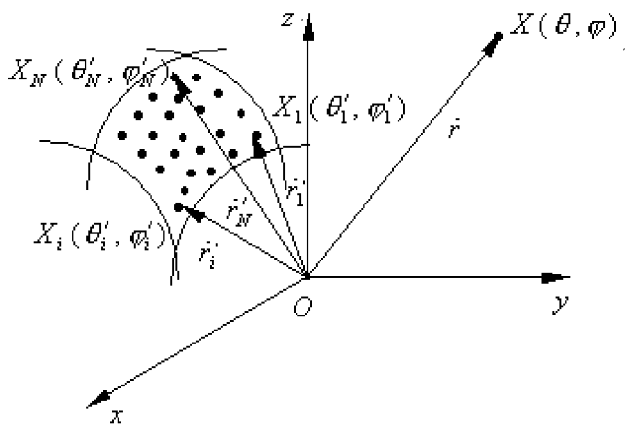


FIGURE 2. Surface multipole expansion

where

$$M_n^m = \frac{4\pi}{2n+1} \sum_{i=1}^{N} q_i \, {r'_i}^n \, Y_{nm}^*(\theta'_i, \varphi'_i) \tag{28}$$

Let $\max_{1 \le i \le N} \{ r'_i \} = \delta_1$. *For every* $p \ge 1$, *the error estimation can be expressed as:*

$$\left| \Phi(X) - \sum_{n=0}^{P} \sum_{m=-n}^{n} \frac{M_n^m}{r^{n+1}} Y_{nm}(\theta, \varphi) \right| \le \left(\frac{\sum_{i=1}^{N} |q_i|}{r - \delta_1} \right) \left(\frac{\delta_1}{r} \right)^{P+1} \tag{29}$$

Proof: Assume that O is the original point, the distance between $X(r, \theta, \varphi)$ and $X_i(r'_i, \theta'_i, \varphi'_i)$ is written as:

$$R_i = |\dot{r} - \dot{r}'_i| = \sqrt{r^2 + {r'_i}^2 - 2 r r'_i \cos\gamma}$$

where $r = |\dot{r}|$, $r'_i = |\dot{r}'_i|$, γ is the included angle of $\dot{r}$ and $\dot{r}'_i$.

Let $t = \frac{r'_i}{r}$, $x = \cos\gamma$. If $|t| < \min \left| x \pm \sqrt{x^2 - 1} \right|$, then

$$\frac{1}{R_i} = \frac{1}{r\sqrt{1 - 2xt + t^2}} = \frac{1}{r} \sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} P_n(x) \frac{{r'_i}^n}{r^{n+1}}$$

From the addition formula expressed by Eq. (12), we have

$$P_n(\cos\gamma) = \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)\, P_n^m(\cos\theta'_i)\, e^{im(\varphi - \varphi'_i)}$$

$$= \sum_{m=-n}^{n} \frac{4\pi}{2n+1} \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta)\, e^{im\varphi} \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta'_i)\, e^{-im\varphi'_i}$$

$$= \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_{nm}(\theta, \varphi)\, Y_{nm}^*(\theta'_i, \varphi'_i)$$

Let $M_n^m = \frac{4\pi}{2n+1} \sum_{i=1}^{N} q_i \, r_i'^{\,n} \, Y_{nm}^*(\theta_i', \varphi_i')$, we have

$$\Phi(X) = \sum_{i=1}^{N} \frac{q_i}{|\dot{r} - \dot{r}_i'|}$$

$$= \sum_{i=1}^{N} q_i \, [\sum_{n=0}^{\infty} (\frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_{nm}(\theta, \varphi) Y_{nm}^*(\theta_i', \varphi_i')) \frac{r_i'^{\,n}}{r^{n+1}}]$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [\frac{4\pi}{2n+1} \sum_{i=1}^{N} q_i \, r_i'^{\,n} \, Y_{nm}^*(\theta_i', \varphi_i')] \frac{1}{r^{n+1}} Y_{nm}(\theta, \varphi)$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{M_n^m}{r^{n+1}} Y_{nm}(\theta, \varphi)$$

It is right Eq. (27). Eq. (29) can be seen in reference [13].

Theorem 4.2. *Assume that there are N charges $X_1, X_2, \cdots, X_N$ with intensity $q_1, q_2, \cdots, q_N$ on surface S, whose spherical polar coordinates are $(r_1', \theta_1', \varphi_1'), (r_2', \theta_2', \varphi_2'), \cdots, (r_N', \theta_N', \varphi_N')$, respectively, as shown in Figure 3. For a random point $X(r, \theta, \varphi) \in R^3$ satisfying $r < \min\limits_{1 \le i \le N} \{r_i'\}$, the potential $\Phi(X)$ generated from charges $q_1, q_2, \cdots, q_N$ is*

$$\Phi(X) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} L_n^m \, Y_{nm}(\theta, \varphi) \, r^n \tag{30}$$

where

$$L_n^m = \frac{4\pi}{2n+1} \sum_{i=1}^{N} q_i \, Y_{nm}^*(\theta_i', \varphi_i') \frac{1}{r_i'^{\,n+1}} \tag{31}$$

Let $\min\limits_{1 \le i \le N} \{r_i'\} = \delta_2$. For every $p \ge 1$, the error estimation can be expressed as:

$$\left| \Phi(X) - \sum_{n=0}^{p} \sum_{m=-n}^{n} L_n^m \, Y_{nm}(\theta, \varphi) \, r^n \right| \le \left(\frac{\sum_{i=1}^{N} |q_i|}{\delta_2 - r} \right) \left(\frac{r}{\delta_2} \right)^{P+1} \tag{32}$$

Proof: Assume that O is the original point, the distance between $X(r, \theta, \varphi)$ and $X_i(r_i', \theta_i', \varphi_i')$ is written as:

$$R_i = |\dot{r} - \dot{r}_i'| = \sqrt{r^2 + r_i'^{\,2} - 2 \, r \, r_i' \cos\gamma}$$

where $r = |\dot{r}|$, $r_i' = |\dot{r}_i'|$, γ is the included angle of $\dot{r}$ and $\dot{r}_i'$.

Let $u = \frac{r}{r_i'}$, $y = \cos\gamma$. If $|u| < \min \left| y \pm \sqrt{y^2 - 1} \right|$, then

$$\frac{1}{R_i} = \frac{1}{r_i' \sqrt{1 - 2\, y\, u + u^2}} = \frac{1}{r_i'} \sum_{n=0}^{\infty} P_n(y) \, u^n = \sum_{n=0}^{\infty} P_n(y) \frac{r^n}{r_i'^{\,n+1}}$$

From the addition formula expressed by Eq. (12), we have

$$P_n(\cos\gamma) = \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)\, P_n^m(\cos\theta_i')\, e^{im(\varphi-\varphi_i')}$$

$$= \sum_{m=-n}^{n} \frac{4\pi}{2n+1} \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta)\, e^{im\varphi} \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta_i')\, e^{-im\varphi_i'}$$

$$= \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_{nm}(\theta,\ \varphi)\, Y_{nm}^*(\theta_i',\ \varphi_i')$$

Let $L_n^m = \frac{4\pi}{2n+1} \sum\limits_{i=1}^{N} q_i\, Y_{nm}^*(\theta_i',\ \varphi_i')\, \frac{1}{{r_i'}^{\,n+1}}$, we have

$$\Phi(X) = \sum_{i=1}^{N} \frac{q_i}{|\dot{r} - \dot{r}_i'|}$$

$$= \sum_{i=1}^{N} q_i \,[\sum_{n=0}^{\infty} (\frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_{nm}(\theta,\ \varphi)\, Y_{nm}^*(\theta_i',\ \varphi_i'))\, \frac{r^n}{{r_i'}^{\,n+1}}]$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [\frac{4\pi}{2n+1} \sum_{i=1}^{N} q_i\, Y_{nm}^*(\theta_i',\ \varphi_i')\, \frac{1}{{r_i'}^{\,n+1}}]\, Y_{nm}(\theta,\ \varphi)\, r^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} L_n^m\, Y_{nm}(\theta,\ \varphi)\, r^n$$

It is right Eq. (30). Eq. (32) can be seen in reference [13].

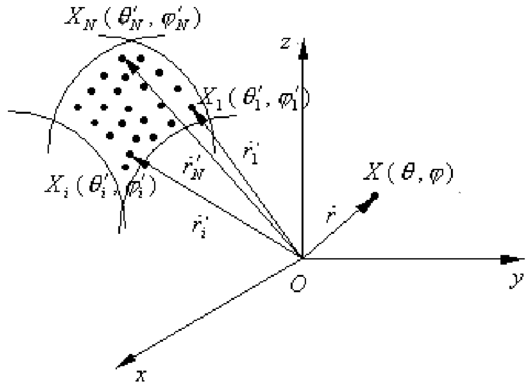


FIGURE 3. Surface local expansion

Assume that $X_i(r_i',\ \theta_i',\ \varphi_i')$ $(i = 1,\ 2,\ \cdots,\ N)$ are random points belonging to different elements on the surface. Let $\max\limits_{1\le i,j\le N} |\dot{r}_i' - \dot{r}_j'| = d_0$, $\min\limits_{1\le k\le N} |\dot{r} - \dot{r}_k'| = d_1$. If $d_1 > 2d_0$, then the surface FM-BEM can be used, as shown in Figure 4.

4.2. **Fundamental theorems for the domain FM-BEM.** Detailed fundamental theorems for the domain FM-BEM have been studied and can be seen in reference [13].

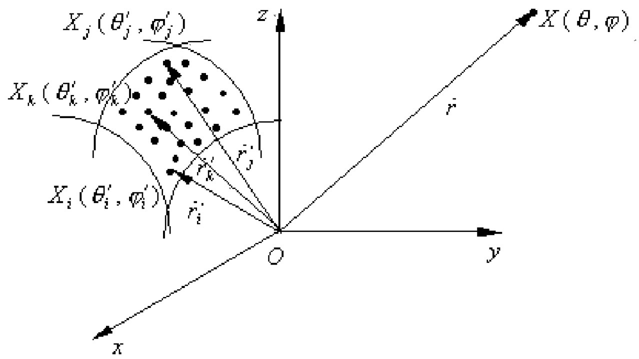
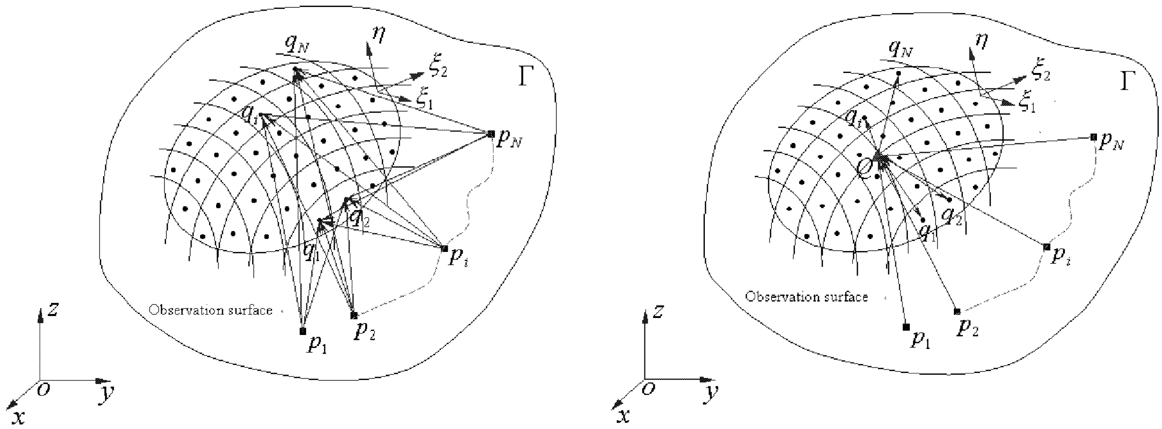


FIGURE 4. Sketch map of the surface FM-BEM

4.3. **Comparison of FM-BEM and conventional BEM for computing influence coefficients.** Compared with conventional BEM, the FM-BEM shows obvious superiority in the computation of influence coefficients, as shown in Figure 5. In Figure 5, q_i denotes the center of the ith element on the observation surface, p_i denotes the ith source point, Q denotes the multipole center.



a) Conventional BEM b) FM-BEM

FIGURE 5. Comparison for computing influence coefficients

From Figure 5, it can be seen that the operations can be approximately reduced from $O(N^2)$ to $O(N)$ if the FM-BEM is used in the computation of influence coefficients. The computational efficiency can be greatly improved. The related numerical example and results analysis will be given in another paper.

5. **Conclusions.** (1) The combination of FMM and BEM was deeply researched. The fundamental formulas were presented for the FM-BEM. (2) Based on current FMM-BEM, a spherical harmonic function and some related FM-BEM numerical formulas were constructed for the boundary surface. (3) The surface FM-BEM theory was established and the FMM-BEM theoretical system was optimized, which laid mathematical foundation for the generalization of FM-BEM in engineering fields. (4) Compared with the conventional

BEM, the presented FM-BEM could reduce the computation of influence coefficients and greatly improve the computational efficiency. However, this advantage could be realized only for large-scale computations.

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