General nonlinear PDE

I. $u(x, y)$:

Quasi-linear PDE:

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

General nonlinear first order PDE:

$$F(x, y, u, u_x, u_y) = 0$$

Second order PDE:

$$au_{xx} + bu_{xy} + u_{yy} + du_x + eu_y + f = 0$$

II. General nonlinear PDE:

$$F(x, y, u, u_x, u_y) = 0$$

III. Given a one-parameter family plane elements, the plane elements envelopes the Monge cone at (x_0, y_0, z_0) it is defined by

$$z - z_0 = p(\lambda)(x - x_0) + q(\lambda)(y - y_0)$$

$$0 = p'(\lambda)(x - x_0) + q'(\lambda)(y - y_0)$$

IV. Assume $z = u(x, y)$ is a solution, it defines a surface

$$z = u(x, y)$$

i.e.

$$z_0 = u(x_0, y_0)$$

let

$$p_0 = u_x(x_0, y_0)$$

$$q_0 = u_y(x_0, y_0)$$

V. The tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = p_0(x - x_0) + q_0(y - y_0)$$

But at (x_0, y_0, z_0), we should satisfy

$$F(x_0, y_0, z_0, p, q) = 0$$

The tangent plane intersect the Monge cone along a line. The line has a direction on the plane, it is called the characteristic direction of the surface

VI. To find the characteristic direction (a, b, c), let $p(\lambda)$ and $q(\lambda)$ be the solution to

$$F(x_0, y_0, z_0, p(\lambda), q(\lambda)) = 0$$

VII.

$$z - z_0 = p(\lambda_0)q'(\lambda_0) - q(\lambda_0)p'(\lambda)$$
$$x - x_0 = -q'(\lambda_0)$$
$$y - y_0 = p'(\lambda_0)$$

Since $F(x_0, y_0, z_0, p, q) = 0$, we have

$$F_p p'(\lambda) + F_q q'(\lambda) = 0$$

Therefore, the characteristic direction at (x_0, y_0, z_0) is

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

where

$$A = F_p$$
$$B = F_q$$
$$C = pF_p + qF_q$$
$$\frac{dx}{dt} = F_p$$
$$\frac{dy}{dt} = F_q$$
$$\frac{du}{dt} = pF_p + qF_q$$

Since $F(x, y, u, u_x, u_y) = 0$, we have

$$F_x + F_u u_x + F_p u_{xx} + F_q u_{yx} = 0$$
$$\frac{dp}{dt} = u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt} = F_p u_{xx} + F_q u_{xy}$$

$$\frac{dx}{dt} = F_p, x(0, s) = x_0(s) \tag{1}$$
$$\frac{dy}{dt} = F_q, x(0, s) = y_0(s) \tag{2}$$
$$\frac{du}{dt} = pF_p + qF_q, u(0, s) = u_0(s) \tag{3}$$
$$\frac{dp}{dt} = -F_x - F_u p, p(0, s) = p_0(s) \tag{4}$$
$$\frac{dq}{dt} = -F_y - F_u q, q(0, s) = q_0(s) \tag{5}$$
$$\tag{6}$$

Figure:

Examples: $u_x^2 + u_y^2 = 1, u_x = u_y$

Find the Monge cone and characteristic line