Second order PDE

I. $u(x,y)$:

II. Second order PDE:

$$a(x,y)u_{xx}+b(x,y)u_{xy}+c(x,y)u_{yy}+du_x+eu_y+qu=p$$

III. Cauchy data

$$u(f(s),g(s))=h(s)$$

$$u_x(f(s),g(s))=l(s)$$

$$u_y(f(s),g(s))=m(s)$$

IV. Strip condition (consistent condition)

$$u_xf'(s)+u_yg'(s)=h'(s)$$

V. A naive way to solve the problem

Since

$$u_x(f(s),g(s))=l(s)$$

we have

$$u_{xx}f'(s)+u_{xy}g'(s)=l'(s)$$

$$u_{xy}f'(s)+u_{yy}g'(s)=m'(s)$$

Combining with

$$a(x,y)u_{xx}+b(x,y)u_{xy}+c(x,y)u_{yy}+du_x+eu_y+qu=p$$

we have

$$\begin{bmatrix} a(x,y) & b(x,y) & c(x,y) \\ f'(s) & g'(s) & 0 \\ 0 & f'(s) & g'(s) \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} dl-em-qh+p \\ l'(s) \\ m'(s) \end{bmatrix}$$

VI. A unique solution can be obtained if

$$det \begin{bmatrix} a(x,y) & b(x,y) & c(x,y) \\ f'(s) & g'(s) & 0 \\ 0 & f'(s) & g'(s) \end{bmatrix} \neq 0$$

VII. Characteristic curves for $f(s)$ and $g(s)$:

$$det \begin{bmatrix} a(x,y) & b(x,y) & c(x,y) \\ f'(s) & g'(s) & 0 \\ 0 & f'(s) & g'(s) \end{bmatrix} = 0$$

A simple form is

$$ag'^2(s)-2bf'g'+cf'^2=0$$