Methods for Solving the First Order PDE

Quasi-linear Form

$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$

Cauchy data $x_0(s),\ y_0(s),\ u_0(s)$

Special case

General Form

$F(x, y, u, u_x, u_y) = 0$

Cauchy data $x_0(s),\ y_0(s),\ u_0(s)$

Method 2

Method of complete integral

Yes

$$\begin{vmatrix} a & b \\ x_0'(s) & y_0'(s) \end{vmatrix} \neq 0$$

No → **Fail**

Yes

Solvability(Fp,Fq) // $(x_0'(s),\ y_0'(s))$

$$\begin{vmatrix} Fp & Fq \\ x_0'(s) & y_0'(s) \end{vmatrix} \neq 0$$

No → **Fail**

Guess $u(x,y)=u(x,y,\alpha,\beta)$

Changing α, β to satisfy *Cauchy data*

$u(x,y)=u(x,y,\alpha(s),\beta(s))$

Method of characteristic curve

$$\frac{dx}{dt} = a(x,y,u),\ x(0,s) = x_0(s)$$
$$\frac{dy}{dt} = b(x,y,u),\ y(0,s) = y_0(s)$$
$$\frac{du}{dt} = c(x,y,u),\ u(0,s) = u_0(s)$$

Special case

Method 1

Method of characteristic strip

$$\frac{dx}{dt} = F_p,\ x(0,s) = x_0(s)$$
$$\frac{dy}{dt} = F_q,\ y(0,s) = y_0(s)$$
$$\frac{du}{dt} = pF_p + qF_q,\ u(0,s) = u_0(s)$$
$$\frac{dp}{dt} = -F_x - pF_u,\ p(0,s) = p_0(s)$$
$$\frac{dq}{dt} = -F_y - qF_u,\ q(0,s) = q_0(s)$$

The $\alpha(s)$、$\beta(s)$ are determined by the equations

$$\begin{cases} u_0(s) = u(x_0(s), y_0(s), a, b) \\ u_0'(s) = u_x x'(s) + u_y y'(s) \end{cases}$$

Basic solution

$u(x,y) = u(x, y, a(s), b(s))$

Singular solution

Envelope (by guessingly of u (x,y))

Where, $p_0(s)$ and $q_0(s)$ are determined from the equations

$$\begin{cases} F(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s)) = 0 \\ u_0'(s) = p_0(s)\,x_0'(s) + q_0(s)\,y_0'(s) \end{cases}$$

Envelope of the family of tangent plane is Monge cone

$$\begin{cases} u - u_0 = p(l)(x - x_0) + q(l)(y - y_0) \\ 0 = p'(l)(x - x_0) + q'(l)(y - y_0) \end{cases}$$