Reflection and transmission in two medium using diamond rule

I. Governing equation

$$u_{xx} = c_1^2 u_{tt}, \quad x > 0$$

$$u_{xx} = c_2^2 u_{tt}, \quad x < 0$$

II. Cauchy data

$$u(x, 0) = 0, \quad x < 0$$

$$u(x, 0) = f(x), \quad x > 0$$

III. Space and time are separated into four regions:

Region I:

$$t > 0$$

$$x - c_1 t > 0$$

Region II:

$$x - c_1 t < 0$$

$$x > 0$$

Region III:

$$x + c_2 t > 0$$

$$x < 0$$

Region VI:

$$x + c_2 t < 0$$

$$t > 0$$

VI. Solution in the four regions:

Region I: $(x_1, t_1) \in I$

$$u^I(x_1, t_1) = 0.5f(x_1 + c_1 t_1) + 0.5f(x_1 - c_1 t_1)$$

Region II: $(x_2, t_2) \in II$

$$u^{II}(x_2, t_2) = 0.5f(x_2 + c_1 t_2) + r(t_2 - \frac{x_2}{c_1}) - 0.5f(c_1 t_2 - x_2)$$

Region III: $(x_3, t_3) \in III$

$$u^{III}(x_3, t_3) = r(t_3 + \frac{x_3}{c_2})$$

Region IV: $(x_4, t_4) \in VI$

$$u^{IV}(x_4, t_4) = 0$$

V. Continuous condition:

Displacement continuous: introduce

$$u(0,t) = r(t)$$

Force equilibrium:

$$T_1 u_x^{II}(0,t) = T_2 u_x^{III}(0,t)$$

where $T_1 = \rho_1 c_1^2$ and $T_2 = \rho_2 c_2^2$.

We have the first order ODE of $r(t)$ as follows:

$$(\rho_1 c_1 + \rho_2 c_2) r'(t) = \rho_1 c_1^2 f'(c_1 t)$$

with the initial condition

$$r(0) = 0$$

The solution of $r(t)$ is

$$r(t) = (\frac{\rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2}) f(c_1 t)$$

Define

$$\kappa = \frac{\rho_2 c_2}{\rho_1 c_1}$$

We have

$$r(t) = \frac{1}{1+\kappa} f(c_1 t)$$

Discussions:

1. Impedance concept.
2. Fixed end: $\kappa \to \infty$, no transmitted.
3. Free end: $\kappa \to 0$, full transmitted.
4. Hopkinson's bar
5. Finite string is also O.K.
6. Extend to two and three dimensional problems
7. Extension to earthquake engineering (compression wave changed to tension wave when free boundary is present)