Clairaut's 微分方程式 奇異解的解法

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Clairaut's 微分方程式:

$$y = xy' - \frac{1}{4}y'^2$$

General solution:

$$y = xc - \frac{1}{4}c^2, for \;\; any \;\; c$$

Singular solution $y = y(x)$ by parameter representation

$$x = x(c), y = y(c)$$

Two conditions must be satisfied if $(x(c), y(c))$ is intersection point with the same tangent line

$$y(c) = x(c)\, c - \frac{1}{4}c^2 \tag{1}$$

$$\frac{dy}{dx}\,|_{(x(c),y(c))} = c$$

By considering

$$\frac{dy}{dx}\,|_{(x(c),y(c))} = \frac{dy(c)/dc}{dx(c)/dc}\,|_{(x(c),y(c))} = c \rightarrow y'(c) = cx'(c)$$

Eq.(1) can be differentiated with respect to c, we have

$$y'(c) = x'(c)\, c + x(c) - \frac{1}{2}c$$

Therefore, we have

$$x(c) = \frac{1}{2}c, y(c) = \frac{1}{4}c^2$$

The singular solution is $y = x^2$.

direct solution for the ODE:

Setting $y' = p$, we have

$$y = xp - \frac{1}{4}p^2$$

$$\frac{dy}{dx} = p + xp' - \frac{1}{4}2p'$$

$$p'(x - \frac{1}{2}p) = 0$$

$$p' = 0 \;\;\rightarrow\;\; p(x) = c \;\;\rightarrow y(x) = cx + k_1$$

$$p = 2x \;\;\rightarrow y(x) = x^2 + k_2$$

where k_1 and k_2 can be determined by substituting into Clairaut's equation.

Exercise: $y = xp - e^p$ where $p = y'$. Solve the general solution and singular solution.