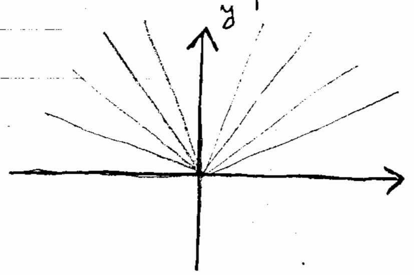
Suppose our equation is for $y \geq 0$



all CHAR cross at the origin
u: singular at $(0,0)$

(singular solution expresses as a weird behavior of the CHAR)

example with e^{-x} - not defined at origine
(identity crisis : value at origine depends on which CHAR we take.)

Generalization:

$a(x,y,z)\, u_x + b(x,y,z)\, u_y + d(x,y,z)\, u_z = 0$
$u = u(x,y,z)$ linear, homogeneous

CHAR : $\dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{dz}{d}$ $\Rightarrow$ u = const along CHAR

$f(x,y,z) = K_1$ $g(x,y,z) = K_2$
↳ surface $\cap$ ↳ surface = curve in 3D space

Case 2 $a(x,y)\, u_x + b(x,y)\, u_y = d(x,y)$
linear , non-homogeneous

same concept: $\Delta x, \Delta y$ $\begin{cases} \Delta x = a \cdot \varepsilon \\ \Delta y = b \cdot \varepsilon \end{cases}$

$\Rightarrow \Delta u \approx u_x \Delta x + u_y \Delta y = (a u_x + b u_y)\, \varepsilon = d\, \varepsilon$

$$\varepsilon = \boxed{\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{d(x,y)}}$$

February 11, 2004 Lecture 3

Review of ODEs: Friday 5-6pm

Review of lecture 2 1st order PDE's

Case 1 $a(x,u)\, u_x + b(x,y)\, u_y = 0$ linear, homogeneous

① CHAR : $\frac{dx}{a} = \frac{dy}{b} \Rightarrow h(x,y) = K_1 = \text{const}$ $\Big\} \Rightarrow u = \overset{\text{arbitrary}}{F}(h(x,y))$ — general solution

② $u = K_2 = \text{const}$ along CHAR

③ Find F by $\boxed{\text{initial data}}$ on curve $\mathcal{Z}$ ← should intersect each CHAR once; (some exceptions)

Case 2 $a(x,y)u_x + b(x,y)u_y = c(x,y)$ linear, non-homogeneous

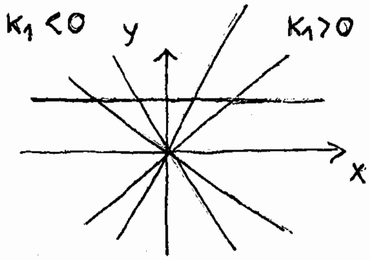
ⓘ CHAR: $\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \Rightarrow h(x,y) = K_1$

(ii) $\frac{du}{c(x,y)} = \frac{dx}{a} = \frac{dx}{b} \Rightarrow f(x,y,u) = K_2$ along CHAR

↗ solve by combining

the difference with Case 1 is that u is not any more a constant along the CHAR

(iii) Apply initial data



Ex.1 IVP $\begin{cases} xu_x + yu_y = 1+y^2 \\ u(x,1) = 1+x \end{cases}$

(i) CHAR: $\frac{dx}{x} = \frac{dy}{y} \Rightarrow y = K_1 x$

(ii) $\frac{du}{1+y^2} = \frac{dx}{x} = \frac{dy}{y}$ combine (1) and (3) $\frac{du}{dy} = \frac{1+y^2}{y} = \frac{1}{y} + y$

$\Rightarrow u = \ln y + \frac{y^2}{2} + K_2$ (this is true along the characteristics!)

$\boxed{u - \ln y - \frac{y^2}{2}} = K_2$

↓ $\bar{u} = K_2$ along CHAR $\Rightarrow K_2 = \bar{u} = F(K_1) = F(\frac{y}{x})$

Map

⇓

$\bar{u} = F(\frac{y}{x})$

$\Rightarrow u - \ln y - \frac{y^2}{2} = F(\frac{y}{x})$

$\boxed{u = \ln y + \frac{y^2}{2} + F(\frac{y}{x})}$ General solution for u

(iii) Apply data

$y=1 \Rightarrow u(x,1) = \overset{0}{\cancel{\ln 1}} + \frac{1}{2} + F(\frac{1}{x}) = 1+x \Rightarrow F(\frac{1}{x}) = \frac{1}{2} + x$

$\Rightarrow F(x) = \frac{1}{2} + \frac{1}{x}$

$\Rightarrow$ $\boxed{u(x,y) = \ln y + \frac{y^2}{2} + \frac{1}{2} + \frac{x}{y}}$ Solution that satisfies initial data

Case 3 $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$ non-linear PDE but linear in u_x, u_y

<u>quasi-linear</u>

$$\begin{cases} \Delta x = a \cdot \varepsilon \\ \Delta y = b \cdot \varepsilon \end{cases} \Rightarrow \Delta u = (a u_x + b u_y)\,\varepsilon = c\,\varepsilon$$

$$\varepsilon = \boxed{\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}}$$

same equation but this time this is a coupled system of ODE's

solutions are $\begin{cases} f(x,y,u) = K_1 \\ g(x,y,u) = K_2 \end{cases}$ (2 eq. with 3 variables)

think of this as 3 independent variables: we have 2 surfaces $\Rightarrow$ curve

$\Rightarrow$ the CHAR are 3D-curves in the $\{x,y,u\}$-space

before, it was curves in the xy-plane

now, the CHAR depend on the form of the solution

Ex. 2 IVP: $\begin{cases} u_t + u u_x = 0 \\ u(x,0) = e^{-x^2} \end{cases}$ Traffic flow

$$\text{CHAR: } \overbrace{\underbrace{\frac{dt}{1} = \frac{dx}{u}}_{} = \frac{du}{0}}^{}$$

with ② over $\frac{dt}{1} = \frac{dx}{u}$ and ① under $\frac{dt}{1} \dots \frac{du}{0}$

① $\Rightarrow$ $\frac{du}{dt} = 0$ $\Rightarrow u = K_1 = \text{const}$ along the CHAR

② $\frac{dx}{u} = \frac{dt}{1}$ not an ODE, we have 3 variables but we know that $u = K_1$ along CHAR

$\Rightarrow \frac{dx}{K_1} = \frac{dt}{1} \Rightarrow x - K_1 t = K_2 = \text{const}$ (! we solve the ODE along CHAR)

$$x - ut = K_2 = \text{const}$$

General solution: to every K_1 we map K_2 or vice versa

$K_2 \xrightarrow{F} K_1$ $K_1 = F(K_2) \Rightarrow \boxed{u = \underbrace{F}_{\text{arbitrary}}(x - ut)}$

- The form of the CHAR in the xy-plane depends on the solution u, not known at this stage.

<u>difference:</u> we have an implicit eqn. for u!

Apply initial data: $u(x,0) = e^{-x^2} = F(x-0) = F(x)$

$\Rightarrow$ $\boxed{u = e^{-(x-ut)^2}}$ that is what we can make, no simpler form solve numerically (interesting phenomena shocks, etc.)

Case 4 $\quad H(x,y,u,\underset{p}{\underset{\downarrow}{u_x}},\underset{q}{\underset{\downarrow}{u_y}}) = 0$ more general case

here we have to view CHAR as curves in space (x,y,u,p,q).

CHAR = a concept: a geometric view of the place where a PDE goes to ODE.

Def:

$p = u_x, \quad q = u_y$

$p_y = q_x$

$$du \approx u_x\,dx + u_y\,dy = p\,dx + q\,dy$$

$$dp = p_x\,dx + p_y\,dy \quad (*)$$

$$dq = q_x\,dx + q_y\,dy \quad (\bigstar)$$

We want to find variations in x & y so that p_x, p_y, q_x, q_y are eliminated from all equations!

In this way, we will get equations only in the variables (x,y,u,p,q)

$$0 = \frac{dH}{dx} = H_x + H_y \overset{0}{\cancel{\frac{dy}{dx}}} + H_u \overset{p}{u_x} + H_p p_x + H_q \overset{p_y}{q_x} = 0$$

$$\Rightarrow \quad H_p p_x + H_q p_y = -(H_x + H_u p) \quad (*)$$ constraint to p_x, p_y

$$0 = \frac{dH}{dy} = H_y + H_x \overset{0}{\cancel{\frac{dx}{dy}}} + H_u \overset{q}{u_y} + H_p \overset{q_x}{p_y} + H_q q_y = 0$$

$$\Rightarrow \quad H_p q_x + H_q q_y = -(H_y + H_u q) \quad (\bigstar)$$ constraint to q_x, q_y

Let's eliminate p_x, p_y from 2(*), and q_x, q_y from 2(★):

$$\begin{cases} dx = H_p \cdot \varepsilon \\ dy = H_q \cdot \varepsilon \end{cases} \Rightarrow \quad du = (pH_p + qH_q)\cdot\varepsilon$$

$$dp = (p_x H_p + p_y H_q)\cdot\varepsilon = -(H_x + H_u p)\cdot\varepsilon$$

$$dq = (q_x H_p + q_y H_q)\cdot\varepsilon = -(H_y + H_y q)\cdot\varepsilon$$

Charpit Eqns.

$$\varepsilon = \boxed{\frac{dx}{H_p} = \frac{dy}{H_q} = \frac{du}{pH_p + qH_q} = \frac{-dp}{H_x + H_u p} = \frac{-dq}{H_y + H_u q}}$$

4 indep. ODE, 5 var.
curve in space (x,y,u,p,q)

$\boxed{\text{Ex. 3}}$ IVP: $\begin{cases} H = p^2 + q + u = 0 \\ u(x,0) = x \end{cases}$ $\quad p \equiv u_x$, $q \equiv u_y$

$H_x = 0$, $H_y = 0$, $H_p = 2p$, $H_q = 1$, $H_u = 1$

Charpit equs:

$$\frac{dx}{2p} = \frac{dy}{1} = \frac{du}{2p^2+q} = -\frac{dp}{p} = -\frac{dq}{q}$$

Choice from where to start

1) $\frac{dx}{2p} = -\frac{dp}{p} \Rightarrow x + 2p = K_1$

2) $-\frac{dp}{p} = -\frac{dq}{q} \Rightarrow (p = K_2 q) \quad q = K_2 p$

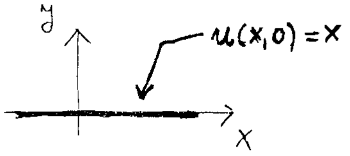
3) $\frac{dy}{1} = -\frac{dp}{p} \Rightarrow pe^y = K_3$

4) $\frac{du}{2p^2+q} = -\frac{dp}{p} \Rightarrow \frac{du}{2p^2+K_2p} = -\frac{dp}{p} \Rightarrow \frac{du}{2p+K_2} = -dp$

$\Rightarrow u = -p^2 - K_2 p + K_4$

One way is to eliminate the constants.

Other way when data is available, use it right away.



parametrize the curve where data is given

$x = s \Rightarrow u(s,0) = s$

goal: express the const of integr. in terms of s

$\begin{cases} p = u_x = u_s = 1 & \text{(direct differentiation)} \\ q = u_y & \text{from the original equation} \quad q = -u - p^2 = -s - 1 \end{cases}$

$$\begin{cases} s + 2\times 1 = K_1 \\ -s - 1 = K_2 \\ 1 = K_3 \\ s + 1 + K_2 = K_4 \end{cases} \Rightarrow \begin{cases} K_1 = s+2 \\ K_2 = -s-1 \\ K_3 = 1 \\ K_4 = s+1-s-1 = 0 \end{cases}$$

influence of data on the const of the solution

$2p = -x + 2 + s$ $\quad p = e^{-y}$, $s = 2e^{-y} + x - 2$, $q = \cdots$

$q = (-s-1)p$

$p = e^{-y}$ $\quad \boxed{u = -e^{-2y} + (2e^{-y} + x - 1)e^{-y}}$

$u = -p^2 + (s+1)p$

x, y, u, p, q, s
$\downarrow$
2 indep. var.