February 4, 2004 Lecture 1

18.306 Advanced PDE with Applications

Dionisios Margetis

ODE

$\boxed{\text{Ex. 1}}$ ODE : $\frac{du}{dx} = 0$, $u = u(x)$

solution is $u = C$, all x

Condition: $u(x_0) = u_0 \Rightarrow C = u_0$

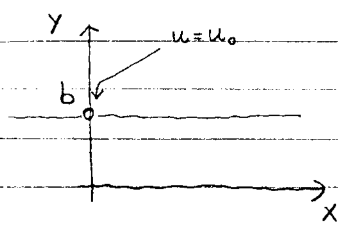
$\Rightarrow u = u_0$ for all x

Conclusion ODE's involve arbitrary constants.

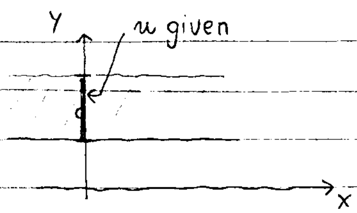
$\boxed{\text{Ex. 2}}$ PDE : $\frac{\partial u}{\partial x} = 0$, $u = u(x, y)$

solution is $u(x,y) = C(y)$ "arbitrary function of y"

General solutions to PDE's involve arbitrary functions.



- by knowing the solution at $(0, b)$ we know the solution at the horizontal line passing through $(0, b)$

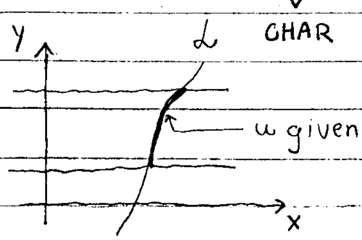


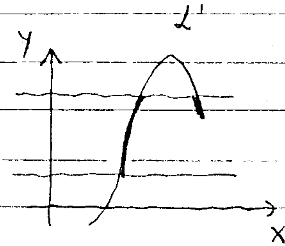
- by knowing u on the segment $y_1 < y < y_2$ we know the solution in the whole strip $y_1 < y < y_2$, all x

$y =$ const : "CHARACTESTIC" by def. line on which u is constant

$\Updownarrow$

CHAR





not possible if u has to be continuous

If u given on $\mathcal{L}'$ then for some y, $y_1 < y < y_2$ I have more than one value of u,

→ not OK unless I allow for "jumps" in u

☆ More, generally, along a CHAR $\boxed{\text{PDE} \Rightarrow \text{ODE}}$ THEME FOR THE COURSE

☆ If a PDE reduces to ODE, it is considered solvable.

Initial Value Problem (IVP) if values of u are given on some open, smooth curve (eg. $\mathcal{L}$)

Generally, PDE is an equ of the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0 \quad \text{where } u = u(x,y)$$

Order of the PDE is the order of the highest derivative of u.

Linear PDE . F is linear function of u and its derivatives.

$\boxed{\text{Ex. 3}}$ PDEs

- $u_t - \overset{\text{const}}{c}\, u_x = 0$ (kinematic equ)
  traffic flow, gas dynamics
- $u_{tt} - c^2 u_{xx} = 0$ (wave equ)
- $u_{xx} + u_{yy} = 0$ (Laplace eqn)
  $\nabla^2 u = 0$

The issue of conditions for PDE

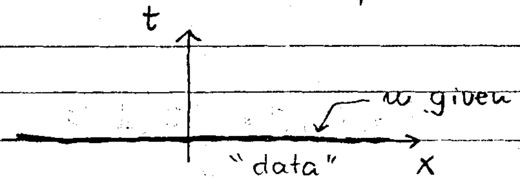
$\boxed{\text{Ex. 4}}$
$$\begin{cases} u'' + u = 0, & u = u(x) \quad 0 \le x \le 1 \\ u(0) = 0 \\ u'(0) = 1 & \text{IVP for ODE} \end{cases}$$

$\boxed{\text{Ex. 5}}$
$$\begin{cases} u'' + u = 0, & u = u(x) \quad 0 \le x \le 1 \\ u(0) = 0 \\ u'(1) = 1 & \text{BVP for ODE} \end{cases}$$

$\boxed{\text{Ex. 6}}$ $u_t - c u_x = 0$, $-\infty < x < +\infty$

$t > 0$

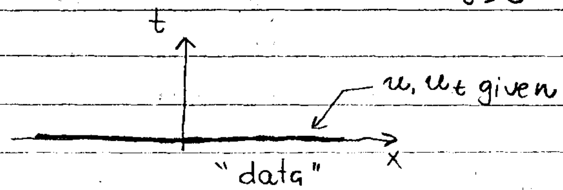
conditions are usually



IVP (there is a unique solution)

$\boxed{\text{Ex. 7}}$ $u_{tt} - c^2 u_{xx} = 0$, $-\infty < x < +\infty$

$t > 0$

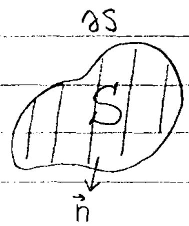


IVP (there is a unique solution)

"Cauchy data"

= u and its normal derivative (u_t) are given on a line

$\boxed{\text{Ex. 8}}$ $\nabla^2 u = 0$ ∂S



(A) u given on ∂S BVP "Dirichlet"

(B) $\frac{\partial u}{\partial n}$ given on ∂S BVP "Neumann"

(C) $au + b\frac{\partial u}{\partial n}$ given on ∂S BVP "Robin" (mixed bc's)

→ for these conditions (ex. 6, 7, 8) solutions not only exist but are unique.

In physical problems, solutions : exist, be unique

$\boxed{\text{Ex. 9}}$ $\begin{cases} u'' = 0 , & 0 < x < 1 \quad u = u(x) \\ u'(0) = 0 , \; u'(1) = 1 \end{cases}$

$u'' = 0 \rightarrow u(x) = Ax + B$, $u'(x) = A$

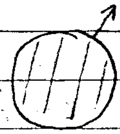
- $u'(0) = 0 \Rightarrow A = 0$
- $u'(1) = 1 \Rightarrow A = 1$

must to be true together ⇒ no solution

IMPOSSIBLE

Alternatively: $0 = \int_0^1 dx\, u''(x) = u'(1) - u'(0) = 1 - 0 = 1$! contradiction

$\boxed{\text{Ex.10}}$ $\nabla^2 u = 0$

BVP:  $\frac{\partial u}{\partial r} = 1$ Neumann bc's

Recall: Divergence theorem $\int_S \nabla \cdot \vec{A}\, d\vec{S} = \int_C \hat{n} \cdot \vec{A}\, dl$

$$0 = \int_S \nabla \cdot \nabla u \, d\vec{S} = \int_C \hat{n} \cdot \nabla u \, dl = \int_O \underbrace{\frac{\partial u}{\partial n}}_{\frac{\partial u}{\partial r}} dl = 1 \times 2\pi r$$

contradiction!

$\Rightarrow$ no solution with this data

ill-posed problems = no solution exists for the given eqn with the given data

$\boxed{\text{Ex.11}}$ $\nabla^2 u = 0$ BVP: $\frac{\partial u}{\partial r} = 0$ no contradiction this time

let u_1 is a solution then $u_2 = u_1 + \text{const}$ is also a solution

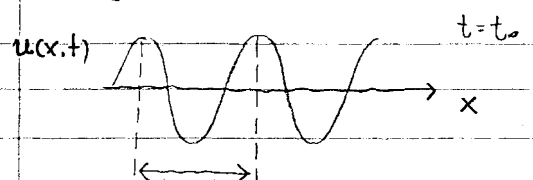
$\Rightarrow$ solution exists but it is not unique

Well-posed problems (classical definition by Hadamard):

1) the solution must exists
2) the solution must be unique
3) the solution depends continuously on data
   (eg. small perturbation of the data results in small perturbation of soln)

Concepts that apply both to linear and nonlinear PDE's

Wave



λ wavelength

$k = \frac{2\pi}{\lambda}$ wavenumber (how many crests/length)

• $u(x,t) = A\cos(\omega t - kx)$, ω frequency (in time)

satisfies the wave equation $u_{tt} - c^2 u_{xx} = 0$, $c = \frac{\omega}{k} = \text{const}$

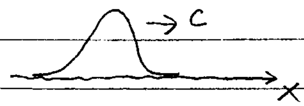
• $u = Ae^{i(\omega t - Kx)}$ satisfies the kinematic equ

$$u_t + c u_x = 0 \qquad c = \frac{\omega}{K} = \text{cont} = \text{"wave speed"}$$

this is an example of solutions (waves) that satisfy some equations.

In general,

$u = f(x-ct)$



bump / perturbation that propagates at speed c

Generalizations of the previous equation:

$u_t + c(u) u_x = 0$

- $c(u) = u \Rightarrow u_t + u u_x = 0$
- $u_t + u u_x = \nu u_{xx}$ Buger's equ.

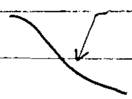
the common concept is a wave-like solution.

February 9, 2004 Lecture 2

Optional Reading Kevorkian § 5.1, 5.2
Debnath § 3.2-3.5

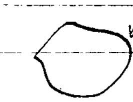
Review of lec. 1:

IVP data on $\mathcal{I}$



$\mathcal{I}$: smooth, open

BVP data on C



C: closed curve

Well-posed problem: ① Solution exists
and ② " is unique
and ③ " depends continuously on data

≠ Ill-posed problem

THEME: $\boxed{\text{PDE} \Rightarrow \text{ODE}}$ both for linear and nonlinear PDE

Useful concept: Wave Simplest PDE: $u_t + c u_x = 0$, $c = \frac{\omega}{K}$ (c: speed, ω: frequency, K: wave #)

(help understand the solution before actually find it)

Modify ↓

$u_t + c(u) u_x = 0$