No. 54 $F(x, y, z, p, q)=0$

例题

1985年 2月～6月

向抄纪都的笔

2020 澳大 1月2日

Ex: Solve

$$\begin{cases} u - xu_x - \frac{1}{2}u_y^2 + x^2 = 0 \\ u(x,0) = x^2 - \frac{1}{6}x^4 \end{cases}$$

$u(s,0) = s^2 - \frac{1}{6}s^4$

write down

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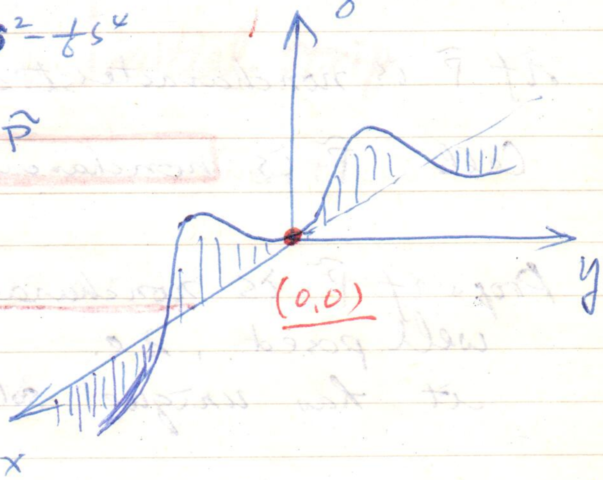
✓ $F(x,y,z,p,q) = z - xp - \frac{1}{2}q^2 + x^2$

$\tilde{\Gamma}$: $x=s$, $y=0$, $h(s) = s^2 - \frac{1}{6}s^4$



① find $p(s)$, $q(s)$ along $\tilde{\Gamma}$

$F(x(s), y(s), z(s), p, q) = 0$ ✓

⊙ $s^2 - \frac{1}{6}s^4 - sp - \frac{1}{2}q^2 + s^2 = 0$

$\frac{dh}{ds} = p\frac{dx}{ds} + q\frac{dy}{ds}$ ✓

$2s - \frac{2}{3}s^3 = p$

$h(s) = u(x(s), y(s))$

∴ $p(s) = 2s - \frac{2}{3}s^3$

⊙ $\frac{dh(s)}{ds} = u_x x'(s) + u_y y'(s)$, $u_x = p$, $u_y = q$

$q(s) = s^2$ or $-s^2$

Say 取 $\begin{cases} p = 2s - \frac{2}{3}s^3 \\ q = s^2 \end{cases}$

$F_p = -x$, $F_q = -q$

☆ $\begin{vmatrix} x'(s) & y'(s) \\ F_p & F_q \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -s & -s^2 \end{vmatrix} = -s^2 \neq 0$ (if $s \neq 0$)

$\frac{dx}{dt} =$, $\frac{dy}{dt} =$, $\frac{dz}{dt} =$, $\frac{dp}{dt} =$, $\frac{dq}{dt} =$

$x(t,$ $q(t,$ $y(t,s$ $p(t,s$ $q(t,$ $z(t,s)$

⇒

$du(x$

for $t=0 \Rightarrow$ initial strip

$$\begin{cases} \frac{dx}{dt} = -x \quad (F_p) \\ \frac{dy}{dt} = -q \quad (F_q) \\ \frac{dz}{dt} = -q(q) + p(-x) = -px - q^2 \quad (pF_p + qF_q) \\ \frac{dp}{dt} = -(-p+2x) - p = -2x \quad -F_x - pF_u \\ \frac{dq}{dt} = -q \quad -F_y - qF_u \end{cases}$$

$$\begin{cases} x(0) = s \\ y(0) = 0 \\ z(0) = s^2 - \frac{1}{4}s^4 \\ p(0) = 2s - \frac{2}{3}s^3 \\ q(0) = s^2 \end{cases}$$

$x = x(s)$

$y = y(s)$

$z = z(s)$

$p = p(s)$

$q = q(s)$

$$\begin{cases} x(t,s) = se^{-t} \quad \text{(ok)} \\ q(t,s) = s^2 e^{-t} \\ y(t,s) = s^2(e^{-t} - 1) \\ p(t,s) = 2se^{-t} - \frac{2}{3}s^3 \\ q(t,s) = s^2 e^{-t} = s^2 e^{-t} \\ z(t,s) = \end{cases}$$

$x = se^{-t}$

$y = s^2(e^{-t} - 1)$

$z =$

Solution

$$\Rightarrow \boxed{u(x,y) = x^2 - x^4/12 + \frac{1}{2}x^2 y + \left(\frac{xy}{3} - \frac{x^3}{12}\right)(x^2 - xy)^{\frac{1}{2}}}, \quad x > 0.$$

$$\frac{du(x(s), y(s))}{ds} = \frac{dh(s)}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

$\frac{\partial u}{\partial x} = p(s)$, $\frac{\partial u}{\partial y} = q(s)$