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Inverse scattering problem for an impedance crack

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Abstract

In this paper we consider an inverse scattering problem whose aim is to recover the impedance function for an arbitrary crack from the far field pattern. Because of the ill-posedness of this problem, regularization method for example, Tikhonov regularization, is incorporated in our solution scheme. Several numerical examples with only one incident wave are given at the end of the paper to show the feasibility of our method.

Keywords: Inverse scattering; Impedance boundary condition; Crack

1. Introduction

The inverse scattering problem which aim is the recovering of the geometry of the obstacles or the investigating of the physical properties of the scatterers has attracted more and more attentions in the past two decades not only because of the pure mathematical interest but also of its applicability in the real world. The inverse scattering problem we are considering in this paper is the problem of finding the impedance of an open arc. The impedance boundary conditions can be used to model practical problems like surface coating which has its application in detection of buried objects, antenna design or the analysis of the earth surface (see [13]). Assuming the crack, our inverse problem of finding the impedance has its application in detection of the corrosion of a pipeline, the flaking or the oxidizing of a wire, for example.

Open arc problem was first investigated by Hayashi [6]. In contrary to the case of a closed boundary, there appeared an integral equation of the first kind instead of a second kind Fredholm integral equation. Thus the Riesz–Fredholm theory was not applicable. Wickham [14] proposed a so called 'crack Green's function' which can transform the problem to an integral equation of the second kind that can be solved by the Riesz–Fredholm theory. The additional difficulty in the treatment of the boundary value problem by an open arc as compared to the case of a closed boundary is that the solution is not smooth. Martin [10] reported that the solution has a square root singularity at the end points of the arc. Using the cosine substitution developed in [15], in the first article on inverse scattering problem from an open arc [7], Kress has overcome this difficulty for a Dirichlet problem. In his paper, integral equation method was used to solve both the direct and inverse problems for

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a sound-soft crack. The scattering problem in the unbounded domain is thus converted into a boundary integral equation. Mönch [11,12] extended this approach to a Neumann crack. Extending to the impedance problem, the method mentioned above did only give less convergence rates in the numerical solution, although it did enable an elegant existence analysis (see [9]). This results from the different structure of the singularities of the solution to the impedance problem at the end points of the crack as compared to those of the Dirichlet or Neumann boundary condition. To be more precise, the solution of the Dirichlet or the Neumann problem has only a square root singularity at the crack tips which can be completely cleared out by the cosine substitution. Besides the square root singularity, the solution of the impedance problem has also a singularity of the logarithmic type at the crack tips (see also [3]) which leads to a slower convergence by the numerics.

For the inverse scattering from an impedance crack, a linear sampling method was used to recover both the unknown crack and the impedance function in [2]. Although the reconstruction of the arc itself is rather successful, the reconstruction of the impedance is not satisfactory. To be more precise, the linear sampling method used in [2] can only reveal the maximal value of the impedance. A pointwise reconstruction of the impedance is not possible. This gives also the cause to this paper.

The plan of the paper is as follows. For the sake of completeness and also the introduction of notations, in Section 2 we will briefly summarize the main results of the direct problem. In Section 3, we will formulate the inverse scattering problem and prove its uniqueness. The notion of the far field operator will be introduced in Section 4. We also describe our solution scheme after the injectivity of the far field operator and the denseness of its range are shown in this section. This will be followed by some numerical examples in the final section.

2. Direct impedance problem

Let $\Gamma \subset \mathbb{R}^2$ be a C^3-smooth open arc, that is, $\Gamma = \{z(s) : s \in [-1,1]\}$ with an injective and three times differentiable parametrization $z : [-1,1] \to \mathbb{R}^2$. The two end points of this open arc will be denoted by $z_{-1} := z(-1)$ and $z_1 := z(1)$. We set $\Gamma_0 := \Gamma \setminus \{z_{-1}, z_1\}$. The orientation of Γ is assumed to be from z_{-1} to z_1. Further we denote by Γ_+ and Γ_- the left- and right-hand sides of Γ, respectively. ν is the unit normal vector to Γ directed toward Γ_+. The mathematical modelling for scattering of time-harmonic acoustic or electromagnetic waves from thin infinitely long cylindrical coated objects leads to the following impedance boundary value problem for the Helmholtz equation in the exterior domain of the crack. The direct scattering problem for an impedance crack that we are considering is the following (Fig. 1).

Problem 1 (*The direct impedance scattering problem*). Find a solution $u^s \in C^2(\mathbb{R}^2 \setminus \Gamma)$ which can be continuously extended to Γ_+ and to Γ_- in the sense that

$$u^s_\pm(x) := \lim_{h\to 0} u^s(x \pm h\nu(x))$$

exists for all $x \in \Gamma$ uniformly.

The solution u^s should satisfy the following conditions:

1. u^s is continuous at the two end points z_{-1}, z_1.
2. $\Delta u^s + k^2 u^s = 0$ in $\mathbb{R}^2 \setminus \Gamma$ with a wave number $k > 0$.
3. The normal derivatives

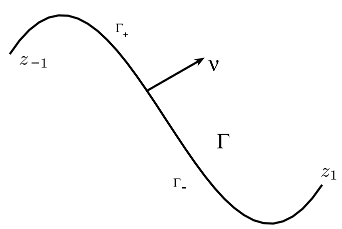


Fig. 1. Open arc.

$$\frac{\partial u^s_\pm(x)}{\partial \nu} := \lim_{h\to+0}\langle \nu(x),\ \text{grad}\ u^s(x\pm h\nu(x))\rangle \tag{1}$$

exist for all $x \in \Gamma_0$ locally uniformly.

4. For $\lambda \in C^{0,\alpha}(\Gamma)$ with $Re(\lambda) \geqslant 0$, it holds the following impedance boundary conditions

$$\frac{\partial u^s_\pm}{\partial \nu} \pm \mathrm{i}k\lambda u^s_\pm = f_\pm \text{ on } \Gamma_0, \tag{2}$$

where $f_\pm \in C^{0,\alpha}(\Gamma)$.

5. u^s satisfies the *Sommerfeld radiation condition*, i.e.,

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u^s}{\partial \nu} - \mathrm{i}ku^s\right) = 0, \qquad r := |x|,$$

uniformly in all directions $\hat{x} := \frac{x}{|x|}$.

We note that we make no assumption on the behavior of the solution at the crack tips except the continuity of the solution. As in the case of a sound-soft crack [7] or in the case of a sound-hard crack [11], the direct impedance problem can be solved by boundary integral equation method. Unlike the two cases mentioned above, our problem is solved with a mixed potential ansatz. In terms of the fundamental solution to the Helmholtz equation in $\mathbb{R}^2$

$$\Phi(x,y) := \frac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|), \qquad x \neq y,$$

we use a combination of a single layer potential and a double layer potential

$$u^s(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi_1(y)\mathrm{d}s(y) + \int_\Gamma \Phi(x,y)\varphi_2(y)\mathrm{d}s(y), \tag{3}$$

with densities $\varphi_1 \in C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma), \varphi_2 \in C(\Gamma)\cap C^{0,\alpha}_{\mathrm{loc}}(\Gamma)$, where

$$C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma) := C^{1,\alpha}_{\mathrm{loc}}(\Gamma_0)\cap\{\varphi \in C(\Gamma)|\varphi(z_{-1}) = \varphi(z_1) = 0, \varphi' \in L^1(\Gamma)\}$$

for $0 < \alpha < 1$. The additional requirement of the local Hölder continuity on φ_2 on Γ_0 serves only to ensure that the ansatz satisfies the boundary conditions (2). For the direct impedance problem, we have the following unique solvability theorem.

Theorem 1. *The direct impedance Problem 1 has a unique solution given by (3) where $\varphi_1 \in C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma), \varphi_2 \in C(\Gamma)\cap C^{0,\alpha}_{\mathrm{loc}}(\Gamma)$ is the (unique) solution to the following system of integral equations*

$$\begin{cases} 2\left(\frac{\partial}{\partial\nu(x)}\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi_1(y)\mathrm{d}s(y) + \int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi_2(y)\right) + ik\lambda(x)\varphi_1(x) = f_-(x) + f_+(x) \\ \varphi_2(x) - 2\mathrm{i}k\lambda(x)\left(\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi_1(y)\mathrm{d}s(y) + \int_\Gamma \Phi(x,y)\varphi_2(y)\mathrm{d}s(y)\right) = f_-(x) - f_+(x) \end{cases} \tag{4}$$

for $x \in \Gamma_0$.

Using the Maue's identity to milden the hypersingularity of the first integral of the first equation in (4), we can rewrite the system (4) in the following form:

$$\begin{cases} \frac{\partial}{\partial\vartheta}S\frac{\partial\varphi_1}{\partial\vartheta} + k^2 < \nu, S\varphi_1\nu > +K'\varphi_2 + \mathrm{i}k\lambda\varphi_1 = f_1 \\ \varphi_2 - \mathrm{i}k\lambda K\varphi_1 - \mathrm{i}k\lambda S\varphi_2 = f_2 \end{cases} \tag{5}$$

with $f_1 := f_+ + f_-$ and $f_2 := f_- - f_+$ and the operators defined by

$$(S\varphi)(x) := 2\int_\Gamma \Phi(x,y)\varphi(y)\mathrm{d}s(y)$$
$$(K\varphi)(x) := 2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\mathrm{d}s(y)$$
$$(K'\varphi)(x) := 2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)\mathrm{d}s(y).$$

With the help of a Green's theorem, the uniqueness of the problem is settled by the Rellich's lemma and the Sommerfeld radiation condition. Using the potential ansatz, the solvability of the boundary value problem is then converted to the solvability of the induced system of two boundary integral Eq. (4) which can be determined by the Riesz theory. For details we refer to [9]. See also [4] for a review on the hypersingularity.

Since in the scattering problem one is concerned about the solution in the exterior unbounded domain, the study of the far-field pattern u_∞ of the scattered field u^s is therefore natural. The far-field pattern describes the behavior of the scattered wave at the infinity

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}}\left\{u_\infty(\widehat{x}) + \mathrm{O}\left(\frac{1}{|x|}\right)\right\} \qquad |x|\to\infty$$

uniformly for all directions $\widehat{x}\in\Omega := \{x\in\mathbb{R}^2 \,|\, |x| = 1\}$. The one-to-one correspondence between radiating waves and their far field patterns is established by the Rellich's lemma.

Using the asymptotic behavior of the Hankel functions, for a plane incident wave we have in the case of a double layer potential the corresponding far field pattern

$$u_{1,\infty}(\widehat{x}) = \frac{1-\mathrm{i}}{4}\sqrt{\frac{k}{\pi}}\int_\Gamma < \nu(y), \widehat{x} > \mathrm{e}^{-\mathrm{i}k<\hat{x},y>}\varphi_1(y)\mathrm{d}s(y) \tag{6}$$

and in the case of a single layer potential the far field pattern

$$u_{2,\infty}(\widehat{x}) = \frac{1+\mathrm{i}}{4\sqrt{k\pi}}\int_\Gamma \mathrm{e}^{-\mathrm{i}k<\hat{x},y>}\varphi_2(y)\mathrm{d}s(y) \tag{7}$$

with the density function φ_1, φ_2 given by Theorem 1. The far field pattern for our impedance problem then reads $u_\infty := u_{1,\infty} + u_{2,\infty}$.

3. Inverse problem for the impedance

In this section, we consider an inverse problem which aim is to recover the unknown impedance function if one knows the crack and the type of the boundary condition. The setting of this inverse problem is meaningful since there exists methods like linear sampling method which can find the crack and the type of the boundary conditions but cannot give a satisfactory pointwise evaluation of the impedance, see for example [2].

Thus we can consider the following inverse problem for impedance

Problem 2 (*The inverse impedance scattering problem*). Assume that an open arc $\Gamma\in\mathbb{R}^2$ with impedance boundary condition is given. The aim of the inverse problem is to reconstruct the unknown impedance function from the knowledge of the (measured) far field pattern $u_\infty(\cdot, d)$ for one incident direction d and for a fixed wave number k.

For this inverse problem, we have the following uniqueness theorem.

Theorem 2. *Assume that λ_1 and λ_2 are two solutions to the inverse impedance problem with the same far field pattern. Then $\lambda_1 = \lambda_2$. In another word, the inverse problem has at most one solution.*

Proof. Assume that λ_1 and λ_2 are two distinct solutions of the inverse problem with the corresponding scattered fields u_1^s and u_2^s which are the solution of the direct problem. After the assumption the scattered field u_1^s and u_2^s have the same far field pattern and therefore must be identical according to the Rellich's lemma, i.e., $u_1^s = u_2^s =: u^s$. For $u := u^s + u^i$ it holds

$$\frac{\partial u_\pm}{\partial v} \pm ik\lambda_1 u_\pm = 0 = \frac{\partial u_\pm}{\partial v} \pm ik\lambda_2 u_\pm \text{ on } \Gamma_0. \tag{8}$$

Now let $\lambda := \lambda_1 - \lambda_2$. Clearly it holds

$$\lambda u_\pm = 0 \text{ on } \Gamma_0.$$

We denote the set $\{x \in \Gamma — \lambda(x) = 0\}$ by Σ. The theorem is then proven if we can show that $\Sigma = \Gamma$. Assume the contrary, $\Sigma \neq \Gamma$. According to the continuity of the impedance λ, there exists a nonempty open subset $\Sigma_0 \subset \Gamma_0$ with the property that $\lambda(x) \neq 0$, for all $x \in \Sigma_0$. This implies that $u_+(x) = u_-(x) = 0$ for all $x \in \Sigma_0$. From the Eq. (8) we have $\frac{\partial u_\pm}{\partial v}(x) = 0, \forall x \in \Sigma_0$. After extending Γ to a closed C^2 – curve, the Holmgren's uniqueness theorem can be applied. Thus we have $u \equiv 0$. This contradicts the fact that the incident wave u^i does not satisfy the Sommerfeld radiation condition. This means that $\lambda \equiv 0$. □

Having the uniqueness, we can now consider the process of the reconstruction. Before that, we have to have something to solve. This will be illustrated in the next section.

4. The far field operator

The task of our inverse problem is to recover the impedance from the far field pattern. This means that we are expecting an equation of the following form

$$\lambda = Au_\infty. \tag{9}$$

The first step in the inverse problem is therefore to define the operator A. Since the Rellich's lemma gives the 1–1 correspondence of the scattered fields and their far field patterns and from the solution theory of the direct problem we know that the scattered field can be written as a function of the densities, it is spontaneous to define the operator $F : L^2(\Gamma) \times L^2(\Gamma) \to L^2(\Omega)$ for $\Psi := (\psi_1, \psi_2) \in L^2(\Gamma) \times L^2(\Gamma)$ through

$$F(\Psi)(\hat{x}) := C_1 \int_\Gamma < v(y), \hat{x} > e^{-ik<\hat{x},y>} \psi_1(y) ds(y) + C_2 \int_\Gamma e^{-ik<\hat{x},y>} \psi_2(y) ds(y). \tag{10}$$

with the constants $C_1 := \frac{1-i}{4}\sqrt{\frac{k}{\pi}}$ and $C_2 := \frac{1+i}{4\sqrt{k\pi}}$.

This operator F, called the far field operator, maps the densities Ψ of the potential u^s to the far field pattern u_∞ of u^s, i.e.,

$$F(\Psi) = u_\infty. \tag{11}$$

Since the kernel of the far field operator is analytic, F is compact. The solving of Ψ from the knowledge of the far field pattern u_∞ is therefore ill-posed (see [8]). In order to solve the far field equation, we have to incorporate some regularization schemes like the Tikhonov regularization. The following theorem justifies the applicability of a regularization scheme.

Theorem 3. *The operator F is injective and has dense range.*

Proof. Assume $F\Psi \equiv 0$. From the definition of the operator F and the Rellich's lemma, we have $u^s(x) = 0$ for all $x \in \mathbb{R}^2 \setminus \Gamma$. As a consequence of the jump relation (in L^2- sense) for the double layer potential, it follows firstly $\varphi_1 \equiv 0$. Then for the single layer potential

$$w(x) := \int_\Gamma \Phi(x, y)\varphi_2(y) ds(y), x \in \mathbb{R}^2 \setminus \Gamma.$$

it holds clearly $w(x) = 0$ for all $x \in \mathbb{R}^2 \setminus \Gamma$. From the jump relation (in L^2- sense) for the single layer potential

$$\lim_{h\to 0}\int_\Gamma\left|\frac{\partial w(x+hv(x))}{\partial v(x)}-\frac{\partial w(x-hv(x))}{\partial v(x)}+\varphi_2(x)\right|^2 \mathrm{d}s(x)=0,$$

we have $\varphi_2 \equiv 0$. We have thus proved the injectivity of the operator F. To prove the second statement of the theorem, it suffices to show that the to F adjoint operator F^* is injective. For $g \in L^2(\Omega)$, the to F adjoint operator $F^* : L^2(\Omega) \to L^2(\Gamma) \times L^2(\Gamma)$ is defined by

$$F^*(g)(y) := \left(\bar{C}_1 \int_\Omega < v(y), \hat{x} > \mathrm{e}^{\mathrm{i}k<\hat{x},y>} g(\hat{x})\mathrm{d}s(\hat{x}), \bar{C}_2 \int_\Omega \mathrm{e}^{\mathrm{i}k<\hat{x},y>} g(\hat{x})\mathrm{d}s(\hat{x})\right), \tag{12}$$

for $y \in \Gamma$. To prove the injectivity of F^*, we follow the idea of the first part of the proof. To this aim, we define for $g \in L^2(\Omega)$ the Herglotz wave function

$$v(y) := \int_\Omega \mathrm{e}^{\mathrm{i}k<\hat{x},y>} g(\hat{x})\mathrm{d}s(\hat{x}), \qquad y \in \mathbb{R}^2.$$

Now assume $F^*(g)(y) = 0$, for $y \in \Gamma$. It follows immediately that $v = 0$ on Γ. For $y \in \Gamma$ it holds further

$$\frac{\partial v(y)}{\partial v(y)} =< v(y),\ \mathrm{grad} v(y) >=< v(y),\ \mathrm{i}kv(y)\hat{x} >= 0.$$

Since the Herglotz wave function v solves the Helmholtz equation, from the Holmgren's uniqueness theorem, we have $v \equiv 0$ in $\mathbb{R}^2$. Since v is a Herglotz wave function, $g \equiv 0$ on Γ (Theorem 3.15 in [5]). It follows the injectivity of F^* and hence the denseness of the range of F. □

Because of the ill-posedness of the far field Eq. (11), we solve the following regularized equation

$$(\alpha I + F^*F)\Psi = F^*u_\infty, \quad \alpha > 0, \tag{13}$$

instead of the original far field Equation (11). From the regularization theory, this equation is uniquely solvable for every positive α. The unique solution of (13), denoted by Ψ_α, interpreted as a minimal norm solution, can be obtained by solving the equation $G(\alpha) = 0$ for a given $\delta > 0$ (see [8]), where the function $G : (0,\infty) \to \mathbb{R}$ is defined by

$$G(\alpha) := \|F\Psi_\alpha - u_\infty\|^2 - \delta^2. \tag{14}$$

After obtaining Ψ, the impedance will be solved from the boundary conditions (2). Indeed, for a fixed plane incident wave we can rewrite the system (5) in the following form

$$\begin{cases} \frac{\partial}{\partial v} K\psi_1 + K'\psi_2 + 2\mathrm{i}k < v, d > u^i = -\mathrm{i}k\lambda(x)\psi_1(x) \\ \mathrm{i}k\lambda(K\psi_1 + S\psi_2 + u^i) = \psi_2. \end{cases} \tag{15}$$

We note here that the second equation in the above system suggests a straight forward calculation formula for the impedance λ, namely

$$\lambda = \frac{\psi_2}{2\mathrm{i}ku} \text{ on } \Gamma_0. \tag{16}$$

We also want to point out that since the total field u cannot vanish on a nonempty open set, it can at most be zero on a set of measure zero. Because of the continuity of the impedance function (or the continuity of the total field), the formula (16) is sensible and also suitable for the numerical realization of the problem.

In terms of the unknown density function $\Psi = (\psi_1, \psi_2)$ we can define the operator

$$R : \Psi \mapsto \frac{\psi_2}{2\mathrm{i}ku}. \tag{17}$$

Thus we can answer the first problem proposed in the beginning of this section, i.e., the operator A defined in (9) is given by

$$A := R(\alpha I + F^*F)^{-1}F^* \tag{18}$$

5. Numerical examples

In this section, we will demonstrate our numerical method through some examples. From the solution theory of the direct scattering problem we see that the scattered field u^s can be represented as a combination of a double layer potential and a single layer potential with different density functions. We choose the solution space for ψ_1 the space

$$T_{1,n} = \text{span}\{\sin x, \sin 2x, \ldots, \sin(n-1)x\}$$

and for ψ_2 the space

$$T_{2,n} = \text{span}\{1, \cos x, \cos 2x, \ldots, \cos nx\}.$$

At this place, we note that although the formula (16) for solving the impedance given in the last section is suitable for the numerics, the solution can be unstable if the total field is very small. Hence we will solve the Eq. (16) in the least square sense. In details we write the impedance function

$$\lambda = \sum_{m=1}^{M} a_m \chi_m \text{ on } \Gamma \tag{19}$$

with unknown coefficients $a_m, m = 1, \ldots, M$ and a set of linear independent functions $\{\chi_1, \chi_2, \ldots, \chi_M\}$. Our task is then to determine the unknowns a_m in the way such that the following sum

$$\sum_{n=1}^{N} \left| \psi_2(x_n) - 2ik u(x_n) \sum_{m=1}^{M} a_m \chi_m(x_n) \right|^2 \tag{20}$$

will be minimized, where $x_n \in \Gamma$, $n = 0, \ldots, N$ are the equidistant knots of the discretization (cf. [1]). At this setting, the number M can also be regarded as a regularization parameter for the inverse problem.

Since no far field data are present for our inverse problem, we have to work with synthetic data. This means that we have to solve the direct problem. The direct problem can be solved by applying the Nyström method to (4). For details we refer to [9]. In order to avoid committing an inverse crime, the number of collocation points used in the inverse solver is chosen to be different from that of the forward solver. In all our examples, we choose 256 equidistant quadrature points for the direct solver which gives the far field pattern at 32 different directions. We note here that as mentioned in the first section, the reason for choosing large number (256) of collocation points for the direct problem is due to the slower convergence rate of the direct impedance problem. For the inverse problem, we choose 32 equidistant quadrature points. The direct reconstruction of the impedance will be denoted by λ_{32}. The parameter M is taken to be 5 and the basis function χ_m is taken to be the trigonometric function $\chi_m(x) := e^{-imx}$ in all the examples. The incident direction is taken to be $(0,1)^t$. The regularization parameter α is determined by trial and error. In all our figures below, the dotted line (black) represents the true solution. We denote by the dashed line (blue) the direct reconstruction using formula (16) and by the solid line (red) the reconstruction with the least square method.

Example 1. For the first example, we take the straight line $y = 0$. On this line we test the impedance functions $\lambda = 0$, 0.5 and $\lambda = \sqrt{1-x^2}$ for the wave number 1. For $\lambda = \sqrt{1-x^2}$ a large wave number $k = 5$ is also tested.

From the results shown in Figs. 2–4, we see that the reconstructions are rather good. Note that the same regularization parameter α is used in all cases. This shows that the choice of α can be independent of the impedance. This is important because the impedance is the unknown for the inverse problem. We also note here that for the case $\lambda = 0$, our impedance problem reduces to a Neumann problem. Theoretically means that we have a better convergence result which is also justified by our numerical reconstruction in Fig. 2.

Example 2. The crack considered in this example is a part of an ellipse,

$$\Gamma = (\cos(t), \sin(t)), \quad t \in [-1, 1]$$

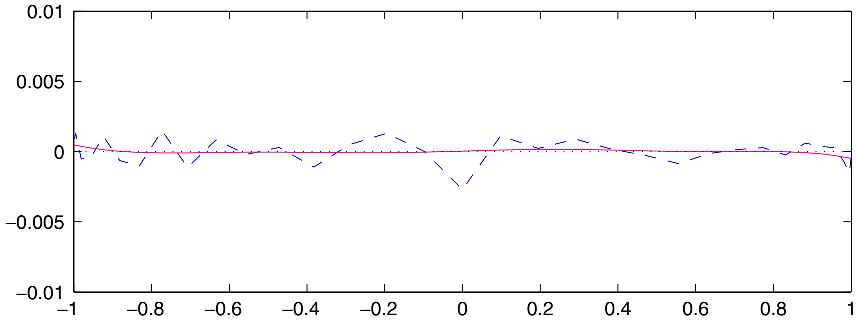


Fig. 2. $k = 1$, $\alpha = 4^{-23}$, $\lambda = 0$, $\|\lambda_{32} - \lambda\|_2 = 2.1\ E^{-4}$.

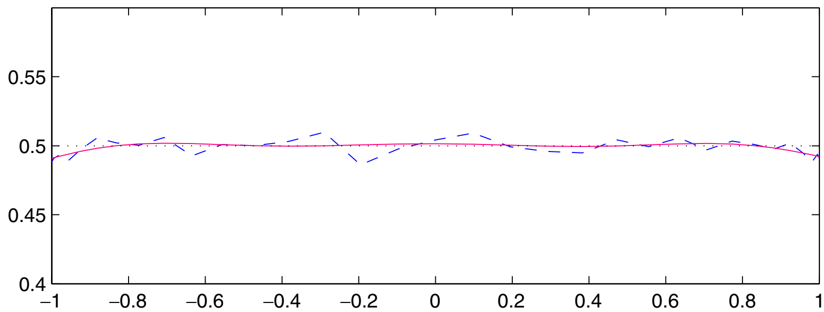
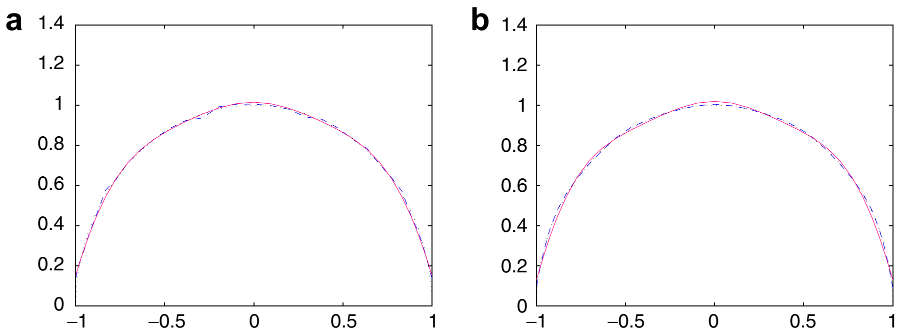


Fig. 3. $k = 1$, $\alpha = 4^{-23}$, $\lambda = 0.5$, $\|\lambda_{32} - \lambda\|_2 = 1.2\ E^{-3}$.



$k = 1, \|\lambda_{32} - \lambda\|_2 = 6.8\,E^{-3}$ $k = 5, \|\lambda_{32} - \lambda\|_2 = 4.4\,E^{-3}$

Fig. 4. $\lambda = \sqrt{1 - x^2}$, $\alpha = 4^{-23}$.

For this example we choose the same regularization parameter as before. The results (Figs. 5 and 6) show that the goodness of the reconstruction is comparable with that of a straight line. We note that for the case of a large wave number, the need of the least square method becomes obvious (Fig. 6b).

Example 3. In this final example, we'd like to show the stability of our method. For this purpose, noisy data will be considered. As test object, we take again the same crack as in the last example. In all the cases, the perturbed far field pattern is taken to be $u_\infty^\delta(\hat{x}_i, d) := \left(1 + \left((-1)^i + \frac{2}{5}\right)\delta\right) u_\infty(\widehat{x}_i, d)$ with $\delta = 5\%$. The results (Figs. 7 and 8) show that our method is stable even when noises are present. Only in the case $k = 5$ with a strongly oscillated incident wave, the direct method yields less accurate reconstruction. However the reconstruction with the aid of least square still gives reasonable result (Fig. 8b).

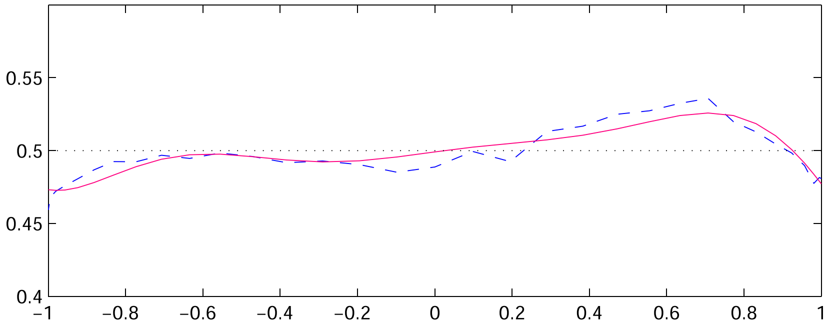


Fig. 5. $k = 1$, $\alpha = 4^{-23}$, $\lambda = 0.5$, $\|\lambda_{32} - \lambda\|_2 = 5.6\ E^{-3}$.

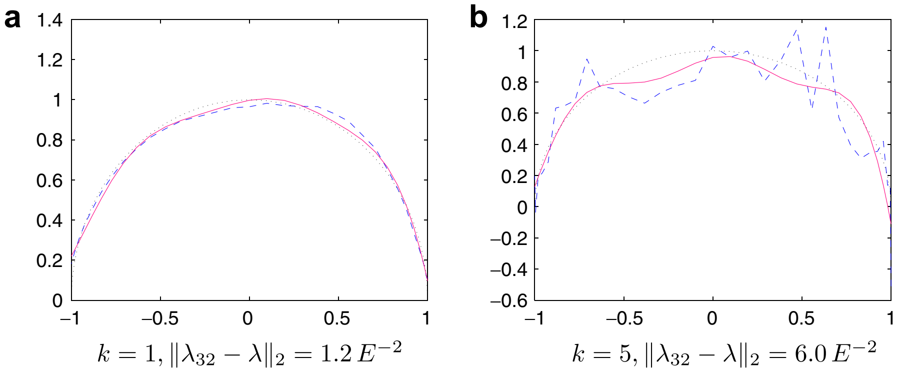


Fig. 6. $\lambda = \sqrt{1 - x^2}$, $\alpha = 4^{-23}$.

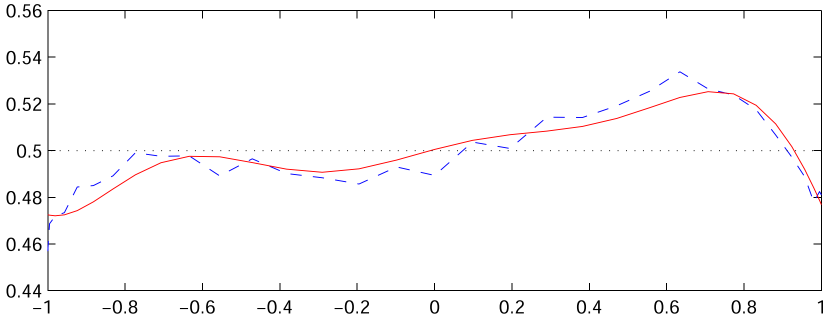


Fig. 7. $k = 1$, $\alpha = 4^{-23}$, $\lambda = 0.5$, $\|\lambda_{32} - \lambda\|_2 = 5.6\ E^{-3}$.

6. Conclusions

The numerical results show among others two major advantages of our solution method. Firstly, our method is direct, simple and fast. Secondly, the choice of the regularization parameter is independent of the unknown impedance. This is very important from the practical viewpoint since one cannot know the impedance in advance. We test a parameter by try and error for the simplest case of a straight line and keep it fixed through all the examples. We also note the parameter α used in our examples are by no means the best possible regularization parameter whose existence is guaranteed by the theory of regularization. This means also that a better result could be achieved by choosing a different α using our algorithm for instance in the very last example. However this is not our intension. From the practical point of view, the detecting of a par-

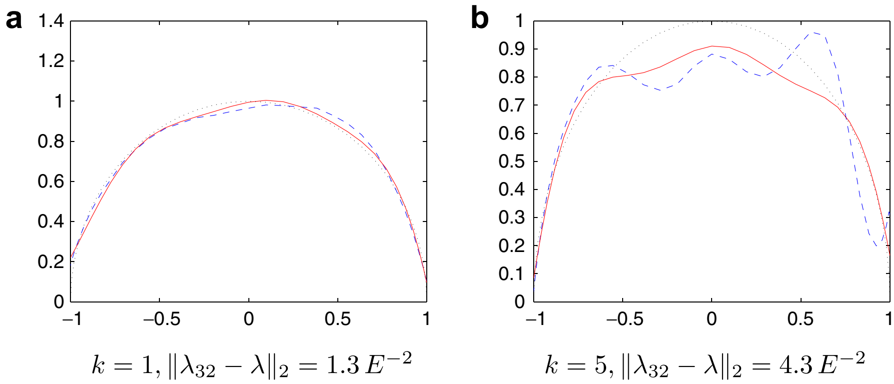


Fig. 8. $\lambda = \sqrt{1 - x^2}$, $\alpha = 4^{-23}$.

tially coated or fully coated object is to determine the unknown coating material, i.e., the impedance function. From the standpoint of the industrial design, one may wish to find the suitable coating which generates the desired pattern of radiation. This all indicates that the algorithm should be independent of the impedance. In particular, the choice of the regularization parameter α should be independent of the configuration. We also note that the method presented in this paper is also stable in the presence of erroneous data.

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