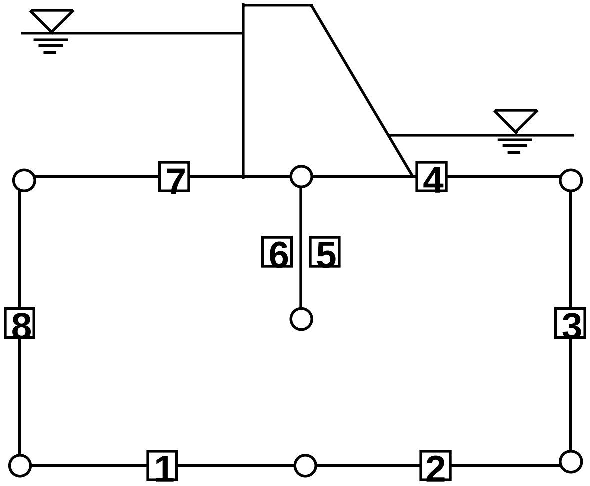
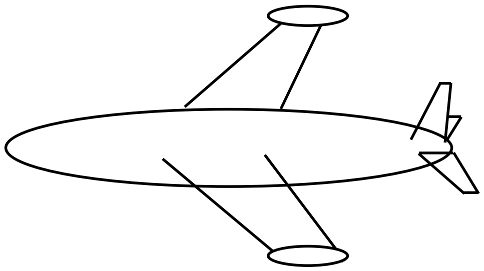
Dual Boundary Element Method and Its Applications

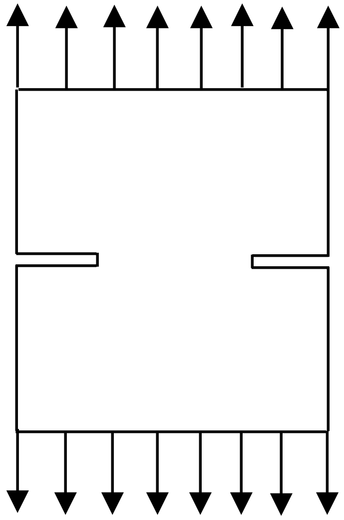
Seepage with sheetpiles



Thin-airfoill Aerodynamics



Crack problem



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Presentation for Feng Chia University

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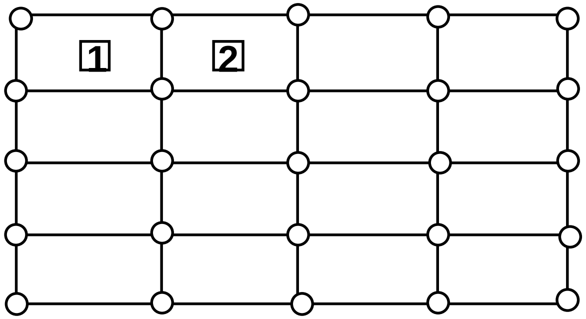
Outline

- Introduction to BEM
- Introduction to dual BEM
- Theory of dual integral equations
- The role of dual integral equations
- Discussion on singular integrals
- Applications
- Conclusions

What Is Boundary Element Method ?

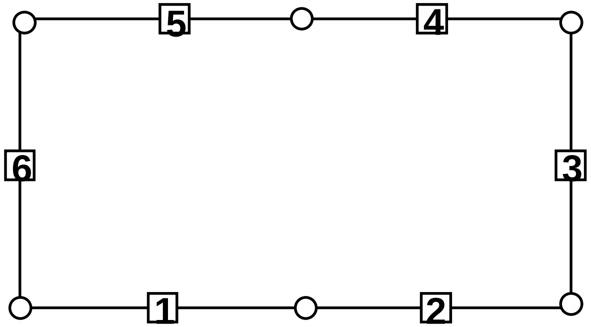
- **Finite element method**



○ **geometry node**

American doctor !

Boundary element method



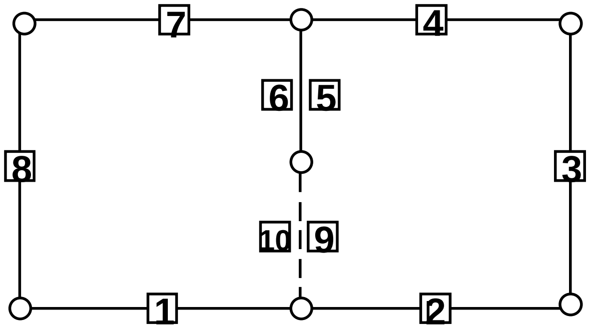
N **the Nth constant or linear element**

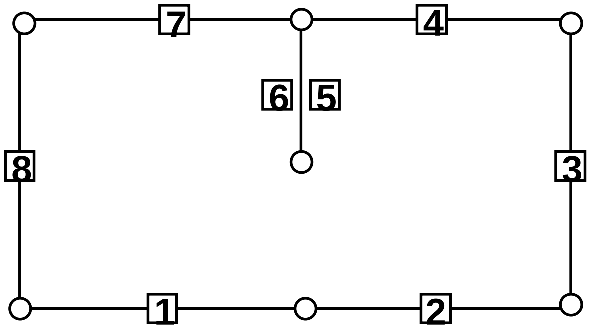
Chinese doctor !

What Is Dual Boundary Element Method ?

- **Boundary element method**

Dual boundary element method



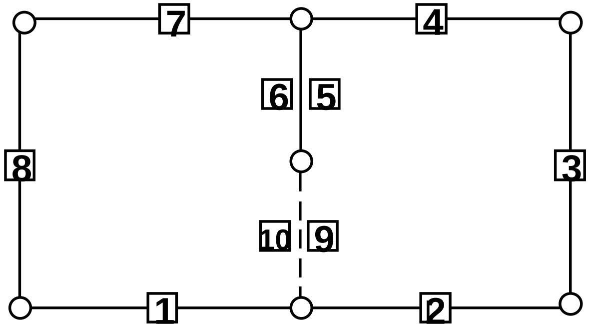


Artifical boundary introduced !

Dual integral equations needed !

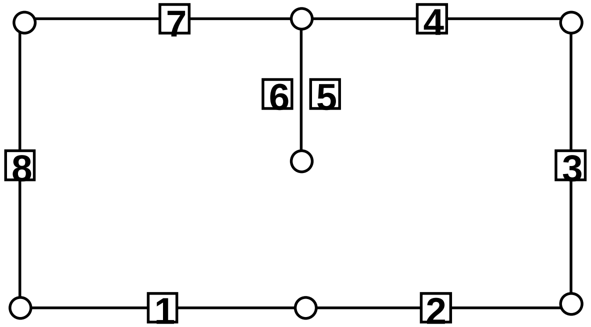
Theory of Dual Integral Equations
Dual Boundary Element Method

- **Boundary element method**



Only U, T equation is used

Dual boundary element method



Both equations are used

$$2\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s), \quad x \in D$$

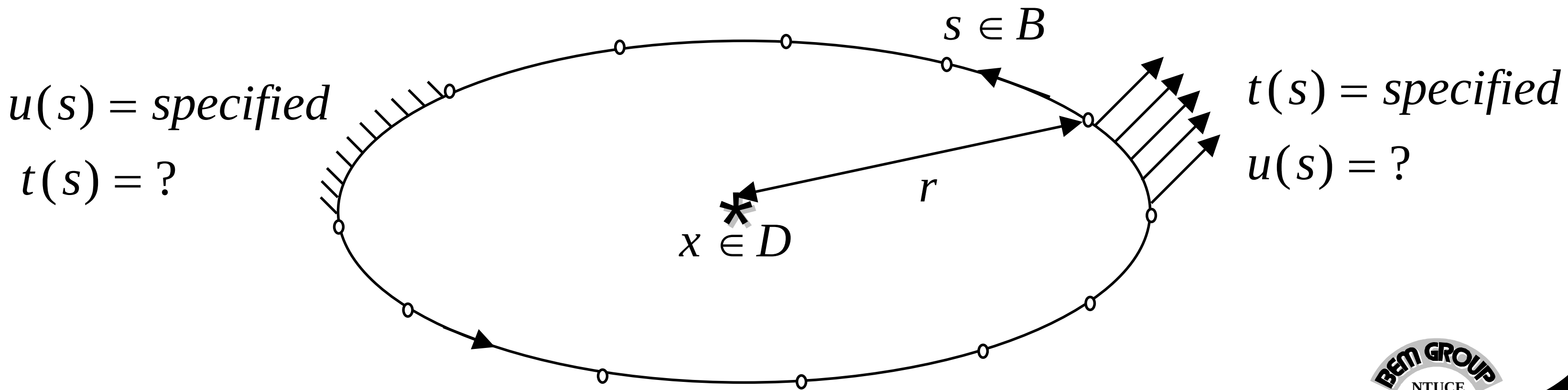
$$2\pi t(x) = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)t(s)dB(s), \quad x \in D$$

Theory of Dual Integral Equations

- **Dual integral equations for domain point**

$$2\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s), \quad x \in D$$

$$2\pi t(x) = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)t(s)dB(s), \quad x \in D$$

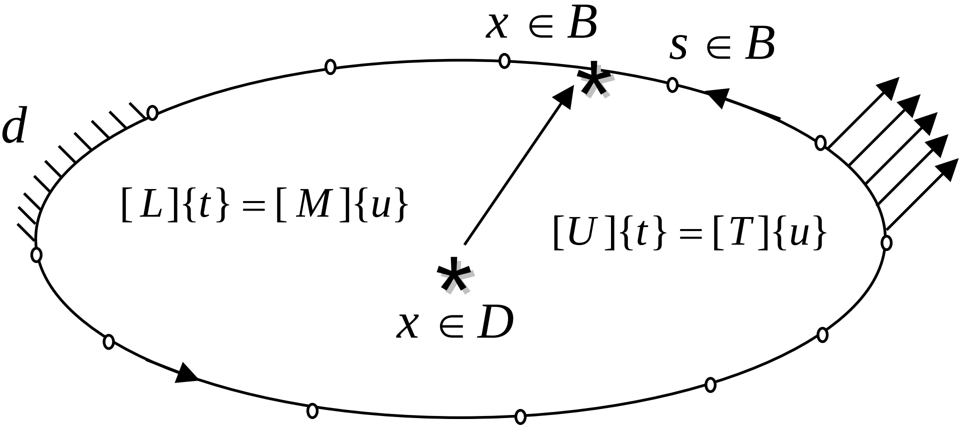


Theory of Dual Integral Equations

- **Dual integral equations for boundary point**

$$\pi u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)t(s)dB(s), \quad x \in B$$

$$\pi t(x) = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)t(s)dB(s), \quad x \in B$$



$x \in B$

$s \in B$

$u(s) = specified$

$t(s) = ?$

$t(s) = specified$

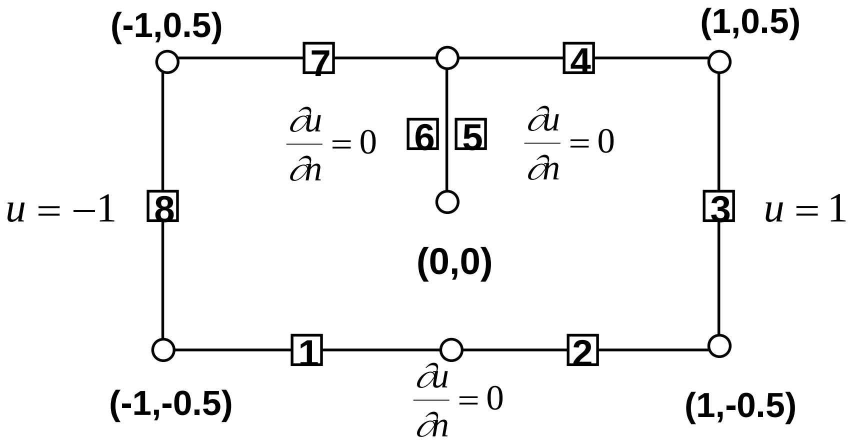
$u(s) = ?$

$[L]\{t\} = [M]\{u\}$

$[U]\{t\} = [T]\{u\}$

$x \in D$

Degeneracy of the Degenerate Boundary



○ **geometry node**

N **the Nth constant or linear element**

$$[U]\{t\}=[T]\{u\}$$

$$[L]\{t\}=[M]\{u\}$$

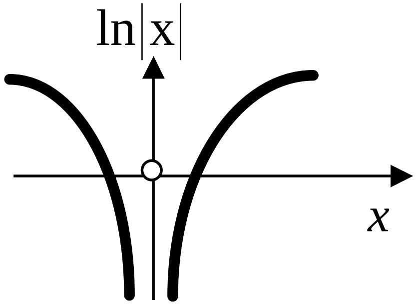
$$[U]=\begin{array}{cccccccc|c}
 & & & & 5(+) & 6(+) & & & \\
\hline
-1.693 & -0.045 & 0.471 & 0.347 & -0.054 & -0.054 & 0.039 & -0.335 & \\
-0.045 & -1.693 & -0.335 & 0.039 & -0.054 & -0.054 & 0.347 & 0.471 & \\
0.445 & -0.335 & -1.693 & -0.335 & 0.019 & 0.019 & 0.445 & 0.703 & \\
0.347 & 0.039 & -0.335 & -1.693 & -0.281 & -0.281 & -0.045 & 0.471 & \\
-0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 & 5(+) \\
-0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 & 6(+) \\
0.039 & 0.347 & 0.471 & -0.045 & -0.281 & -0.281 & -1.693 & -0.334 & \\
-0.335 & 0.445 & 0.703 & 0.445 & 0.019 & 0.019 & -0.335 & -1.693 &
\end{array}$$

$$[T]=\begin{array}{cccccccc|c}
n(s) & & & & 5(+) & 6(-) & & & \\
\hline
-\pi & 0.000 & 0.588 & 0.519 & -0.321 & 0.321 & 0.927 & 1.107 & \\
0.000 & -\pi & 1.107 & 0.927 & 0.321 & -0.321 & 0.519 & 0.588 & \\
0.219 & 1.107 & -\pi & 1.107 & 0.464 & -0.464 & 0.219 & 0.490 & \\
0.519 & 0.927 & 1.107 & -\pi & 0.785 & -0.785 & 0.000 & 0.588 & \\
0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 & 5(+) \\
0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 & 6(+) \\
0.927 & 0.519 & 0.588 & 0.000 & -0.7854 & 0.785 & -\pi & 1.107 & \\
1.107 & 0.219 & 0.490 & 0.219 & -0.464 & 0.464 & 1.107 & -\pi &
\end{array}$$

$$[L]=\begin{array}{c|cccccccc|c}
 & & & & & 5(+) & 6(+) & & & \\
\hline
n(x) & \pi & 0.000 & 0.184 & 0.519 & 0.458 & 0.458 & 0.927 & 0.805 & \\
 & 0.000 & \pi & 0.805 & 0.927 & 0.458 & 0.458 & 0.519 & 0.184 & \\
 & 0.612 & 0.805 & \pi & 0.805 & 0.464 & 0.464 & 0.612 & 0.490 & \\
 & 0.519 & 0.927 & 0.805 & \pi & 0.347 & 0.347 & 0.000 & 0.184 & \\
 & -0.511 & 0.511 & 0.888 & 1.417 & \pi & -\pi & -1.417 & -0.888 & 5(+) \\
 & 0.511 & -0.511 & -0.888 & -1.417 & -\pi & \pi & 1.417 & 0.888 & 6(-) \\
 & 0.927 & 0.519 & 0.184 & 0.000 & 0.347 & 0.347 & \pi & 0.805 & \\
 & 0.805 & 0.612 & 0.490 & 0.612 & 0.464 & 0.464 & 0.805 & \pi &
\end{array}$$

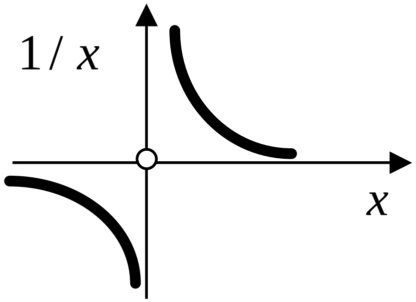
$$[M]=\begin{array}{c|cccccccc|c}
 & n(s) & & & & 5(+) & 6(-) & & & \\
\hline
n(x) & 4.000 & -1.333 & -0.205 & -0.061 & 0.600 & -0.600 & -0.800 & -1.600 & \\
 & -1.333 & 4.000 & -1.600 & -0.800 & -0.600 & 0.600 & -0.061 & -0.205 & \\
 & -0.282 & -1.600 & 4.000 & -1.600 & -0.400 & 0.400 & -0.282 & -0.236 & \\
 & -0.061 & -0.800 & -1.600 & 4.000 & -1.000 & 1.000 & -1.333 & -0.205 & \\
 & 0.853 & -0.853 & -0.715 & -3.765 & 8.000 & -8.000 & 3.765 & 0.715 & 5(+) \\
 & -0.853 & 0.853 & 0.715 & 3.765 & -8.000 & 8.000 & -3.765 & -0.715 & 6(-) \\
 & -0.800 & -0.062 & -0.205 & -1.333 & 1.000 & -1.000 & 4.000 & -1.600 & \\
 & -1.600 & -0.282 & -0.235 & -0.282 & 0.400 & -0.400 & -1.600 & 4.000 &
\end{array}$$

Definitions of R.P.V., C.P.V. and H.P.V.



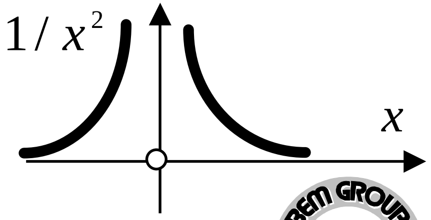
- **R.P.V.**

$$R.P.V.\int_{-1}^{1} \ln|x|\ dx = (x\ln|x| - x)\ \Big|_{x=-1}^{x=1} = -2$$



- **C.P.V.**

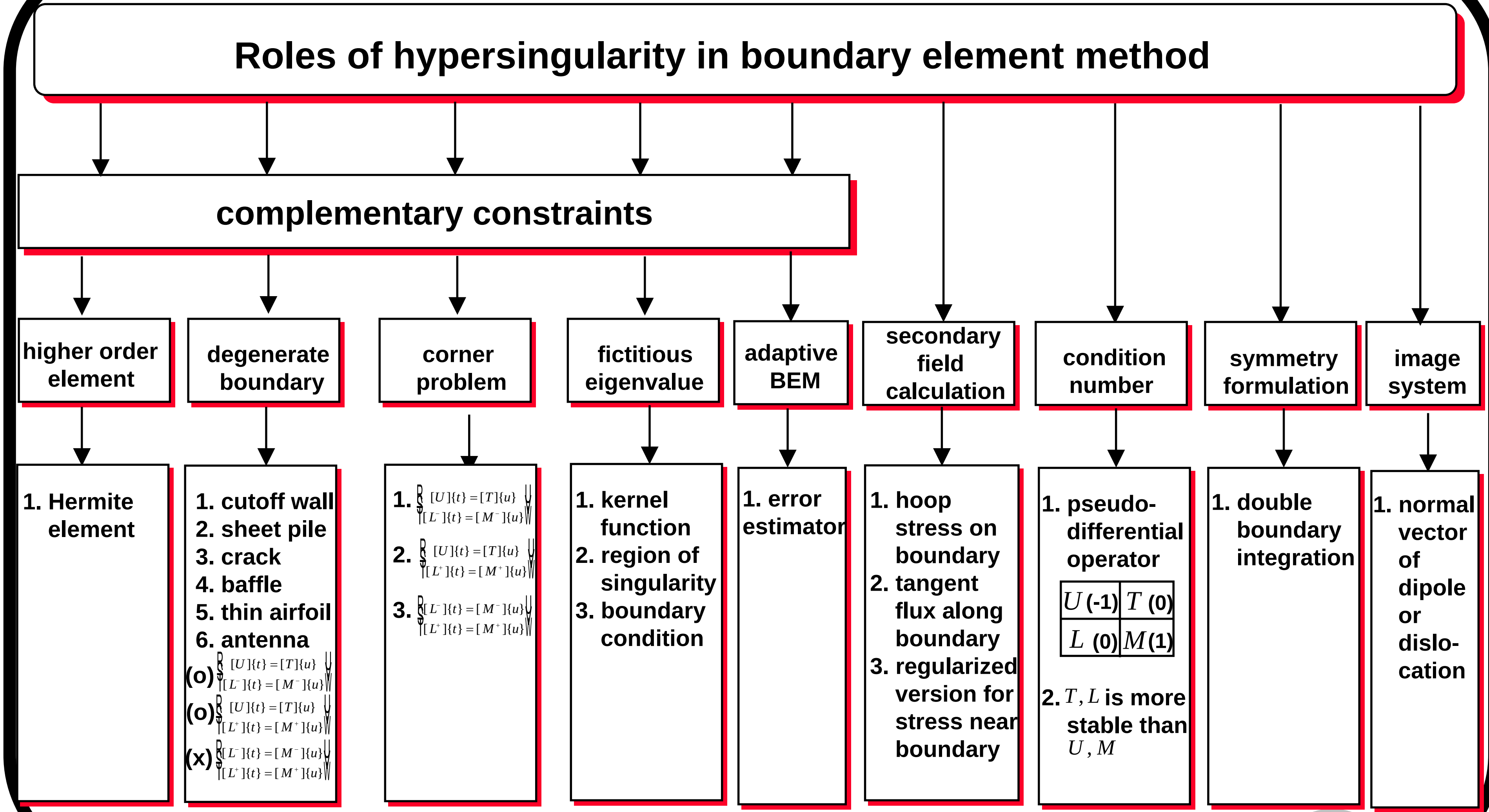
$$C.P.V.\int_{-1}^{1} \frac{1}{x}\ dx = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{1}{x}\ dx = 0$$



- **H.P.V.**

$$H.P.V.\int_{-1}^{1} \frac{1}{x^2}\ dx = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{1}{x^2}\ dx - \frac{2}{\varepsilon} = -2$$

Roles of hypersingularity in boundary element method



complementary constraints

higher order element

1. Hermite element

degenerate boundary

1. cutoff wall
2. sheet pile
3. crack
4. baffle
5. thin airfoil
6. antenna

(o) $\left\{ \begin{array}{l} [U]\{t\}=[T]\{u\} \\ [L^-]\{t\}=[M^-]\{u\} \end{array} \right\}$

(o) $\left\{ \begin{array}{l} [U]\{t\}=[T]\{u\} \\ [L^+]\{t\}=[M^+]\{u\} \end{array} \right\}$

(x) $\left\{ \begin{array}{l} [L^-]\{t\}=[M^-]\{u\} \\ [L^+]\{t\}=[M^+]\{u\} \end{array} \right\}$

corner problem

1. $\left\{ \begin{array}{l} [U]\{t\}=[T]\{u\} \\ [L^-]\{t\}=[M^-]\{u\} \end{array} \right\}$
2. $\left\{ \begin{array}{l} [U]\{t\}=[T]\{u\} \\ [L^+]\{t\}=[M^+]\{u\} \end{array} \right\}$
3. $\left\{ \begin{array}{l} [L^-]\{t\}=[M^-]\{u\} \\ [L^+]\{t\}=[M^+]\{u\} \end{array} \right\}$

fictitious eigenvalue

1. kernel function
2. region of singularity
3. boundary condition

adaptive BEM

1. error estimator

secondary field calculation

1. hoop stress on boundary
2. tangent flux along boundary
3. regularized version for stress near boundary

condition number

1. pseudo-differential operator

U (-1)	T (0)
L (0)	M (1)

2. T, L is more stable than U, M

symmetry formulation

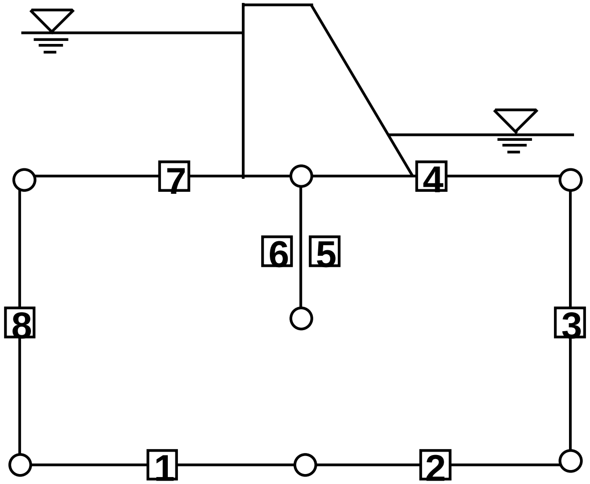
1. double boundary integration

image system

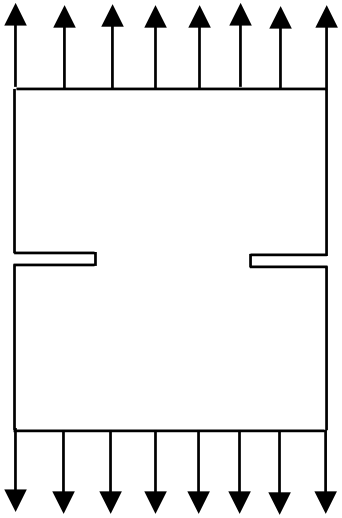
1. normal vector of dipole or dislocation

Applications of Dual Integral Equations

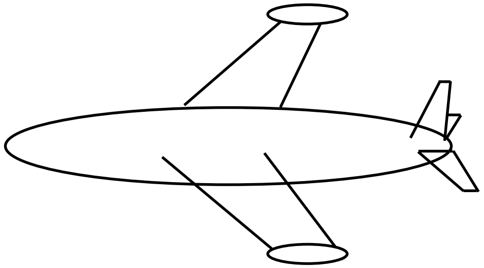
Seepage with sheetpiles



Crack problem



Thin-airfoil Aerodynamics



Conclusions

- **The theory of dual integral equation has been introduced**
- **The role of hypersingularity is examined**
- **The dual boundary element program has been implemented**
- **The applications to seepage flow with sheet piles, crack problem and thin airfoil aerodynamics have been demonstrated.**

Hypersingularity → Divergent series