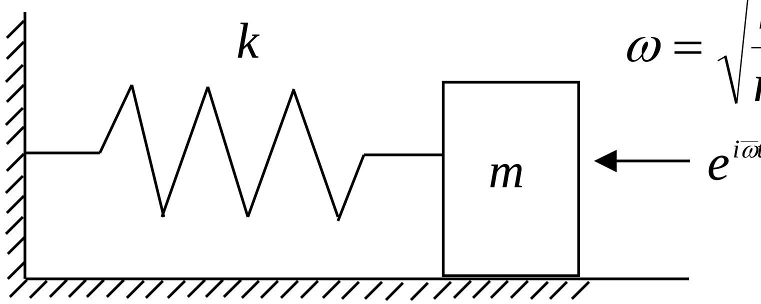
On the Mechanism of Fictitious Eigenvalues in Direct and Indirect BEM

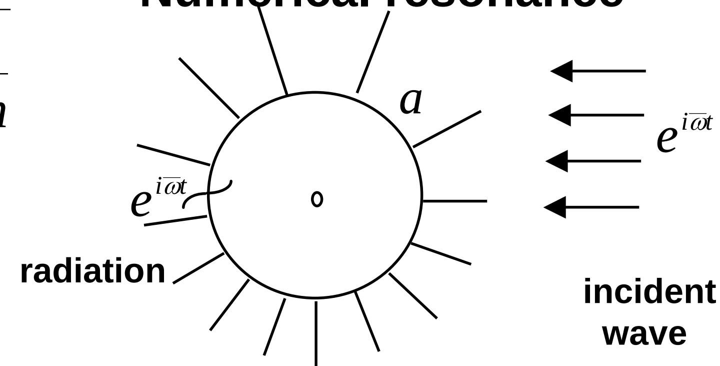
Physical resonance

$$\omega = \sqrt{\frac{k}{m}}$$



$$u = \frac{finite}{(\omega^2 - \bar{\omega}^2)} \to \infty, \text{ if } \bar{\omega} \to \omega$$

Numerical resonance



$$u = \lim_{\bar{\omega} \to \omega} \frac{0}{0} \to finite, \text{ if } \bar{\omega} \to \omega$$

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National Taiwan University

Presentation for BEM Course

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(FICT2D.PPT)

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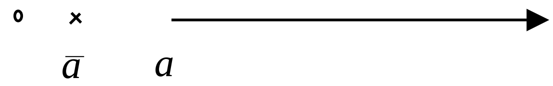
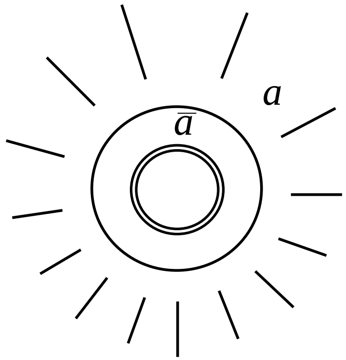
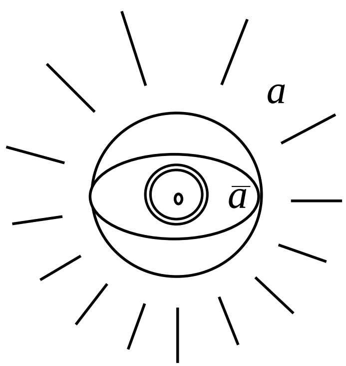


one dimensional problem



two dimensional problem



three dimensional problem

Indirect method

1-D	$\bar{u}=\bar{u}_0$	$\bar{t}_0=\bar{t}_0$
U, L	$\cos(k\bar{a})=0$	$\cos(k\bar{a})=0$
T, M	$\sin(k\bar{a})=0$	$\sin(k\bar{a})=0$

2-D	$\bar{u}=\bar{u}_0$	$\bar{t}_0=\bar{t}_0$
U, L	$J_n(k\bar{a})=0$	$J_n(k\bar{a})=0$
T, M	$J_n'(k\bar{a})=0$	$J_n'(k\bar{a})=0$

3-D	$\bar{u}=\bar{u}_0$	$\bar{t}_0=\bar{t}_0$
U, L	$j_n(k\bar{a})=0$	$j_n(k\bar{a})=0$
T, M	$j_n'(k\bar{a})=0$	$j_n'(k\bar{a})=0$

Direct method

1-D	$\bar{u}=\bar{u}_0$	$\bar{t}_0=\bar{t}_0$
U, T	$\cos(ka)=0$	$\cos(ka)=0$
L, M	$\sin(ka)=0$	$\sin(ka)=0$

2-D	$\bar{u}=\bar{u}_0$	$\bar{t}_0=\bar{t}_0$
U, T	$J_n(ka)=0$	$J_n(ka)=0$
L, M	$J_n'(ka)=0$	$J_n'(ka)=0$

3-D	$\bar{u}=\bar{u}_0$	$\bar{t}_0=\bar{t}_0$
U, T	$j_n(ka)=0$	$j_n(ka)=0$
L, M	$j_n'(ka)=0$	$j_n'(ka)=0$

Fig.6.10 Fictitious eigenvalues using different methods

Degenerate Form for Kernel Functions

interior S **exterior**

interior

$$U^i(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}C_m(ks)R_m(kx)\qquad \ominus\quad x\in D^i$$

$$T^i(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}\{\nabla_s C_m(ks)\cdot n(s)\}\underline{R_m(kx)}$$

$$L^i(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}C_m(ks)\underline{\{\nabla_x R_m(kx)\cdot n(x)\}}$$

$$M^i(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}\{\nabla_s C_m(ks)\cdot n(s)\}\{\nabla_x R_m(kx)\cdot n(x)\}$$

jump

jump

exterior

$$U^e(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}C_m(kx)\underline{R_m(ks)}\qquad \ominus\quad x\in D^e$$

$$T^e(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}C_m(kx)\{\nabla_s R_m(ks)\cdot n(s)\}$$

$$L^e(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}\{\nabla_x C_m(kx)\cdot n(x)\}R_m(ks)$$

$$M^e(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}\{\nabla_x C_m(kx)\cdot n(x)\}\underline{\{\nabla_s R_m(ks)\cdot n(s)\}}$$

item / case	$C_m(kx)=R_m(kx)-i\,I_m(kx)$	$R_m(ks)$	$I_m(ks)$	c_m
1-D rod	e^{-ikx}	$\cos(ks)$	$\sin(ks)$	k
2-D disc	$H_m^{(2)}(k\rho)\,e^{-im\theta}$	$J_m(k\bar{\rho})\,e^{im\bar{\theta}}$	$Y_m(k\bar{\rho})\,e^{im\bar{\theta}}$	4
3-D sphere	$h_m^{(2)}(k\rho)P_m^p(\cos\theta)\cos(p\phi)$	$j_m(k\bar{\rho})P_m^p(\cos\bar{\theta})\cos(p\bar{\phi})$	$y_m(k\bar{\rho})P_m^p(\cos\bar{\theta})\cos(p\bar{\phi})$	$4\pi/k$

Degenerate forms of kernel function

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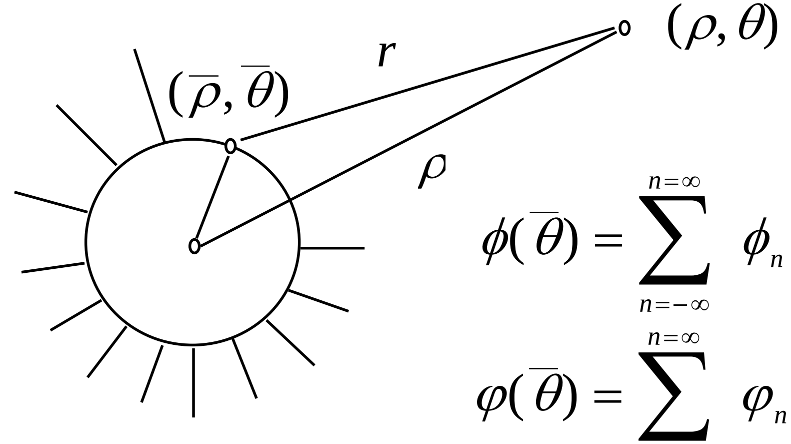
$\rho > \bar{\rho}$ $\quad x = (\rho, \theta) \quad s = (\bar{\rho}, \bar{\theta})$

$$(\nabla^2 + k^2)\, u = 0$$



$$U(s,x) = H_0^{(1)}(kr)$$

$$= \sum_{n=-\infty}^{n=\infty} J_n(k\bar{\rho})\, H_n^{(1)}(k\rho)\, e^{in(\bar{\theta}-\theta)}$$

$$T(s,x) = \sum_{n=-\infty}^{n=\infty} k\, J_n'(k\bar{\rho})\, H_n^{(1)}(k\rho)\, e^{in(\bar{\theta}-\theta)}$$

$$L(s,x) = \sum_{n=-\infty}^{n=\infty} k\, J_n(k\bar{\rho})\, H_n^{(1)}{}'(k\rho)\, e^{in(\bar{\theta}-\theta)}$$

$$M(s,x) = \sum_{n=-\infty}^{n=\infty} k^2\, J_n'(k\bar{\rho})\, H_n^{(1)}{}'(k\rho)\, e^{in(\bar{\theta}-\theta)}$$

$$\phi(\bar{\theta}) = \sum_{n=-\infty}^{n=\infty} \phi_n\, e^{-in\bar{\theta}}$$

$$\varphi(\bar{\theta}) = \sum_{n=-\infty}^{n=\infty} \varphi_n\, e^{-in\bar{\theta}}$$

$$u(\theta) = \sum_{n=-\infty}^{n=\infty} u_n\, e^{-in\theta}$$

$$t(\theta) = \sum_{n=-\infty}^{n=\infty} t_n\, e^{-in\theta}$$

$$u(\rho,\theta) = \int_{B(s)} U(s,x)\,\phi(s)\,dB(s) - \int_{B(s)} T(s,x)\,\varphi(s)\,dB(s)$$

$$t(\rho,\theta) = \int_{B(s)} L(s,x)\,\phi(s)\,dB(s) - \int_{B(s)} M(s,x)\,\varphi(s)\,dB(s)$$

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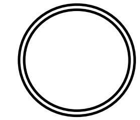
$$\sum_{n=-\infty}^{n=\infty} u_n \, e^{-in\theta} = \sum_{n=-\infty}^{n=\infty} U_n \, \phi_n \, e^{-in\theta} - \sum_{n=-\infty}^{n=\infty} T_n \, \varphi_n \, e^{-in\theta}$$

$$\sum_{n=-\infty}^{n=\infty} t_n \, e^{-in\theta} = \sum_{n=-\infty}^{n=\infty} L_n \, \phi_n \, e^{-in\theta} - \sum_{n=-\infty}^{n=\infty} M_n \, \varphi_n \, e^{-in\theta}$$

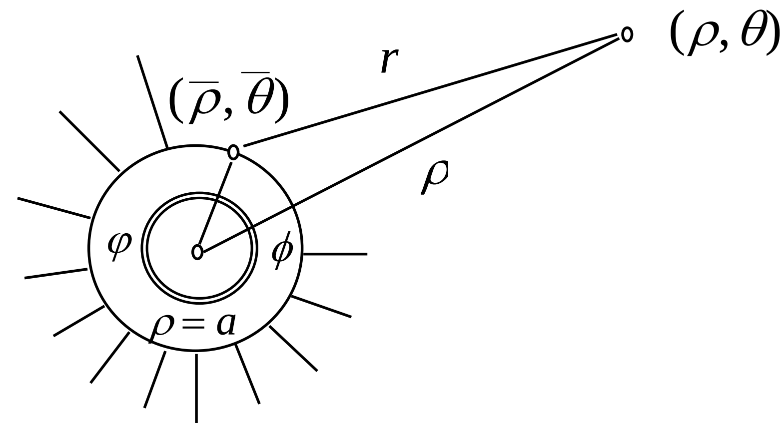


singularity ϕ φ
distribution region

Field representation: (Indirect method)

$$u_n(\rho) = U_n(\rho)\,\phi_n - T_n(\rho)\,\varphi_n$$

$$t_n(\rho) = L_n(\rho)\,\phi_n - M_n(\rho)\,\varphi_n$$



Boundary representation:

$\rho \to \bar{\rho}$

$$u_n(\bar{\rho}) = U_n(\bar{\rho})\,\phi_n - T_n(\bar{\rho})\,\varphi_n$$

$$t_n(\bar{\rho}) = L_n(\bar{\rho})\,\phi_n - M_n(\bar{\rho})\,\varphi_n$$

Field representation: (Direct method)

$$u_n(\rho) = U_n(\rho)\,t_n - T_n(\rho)\,u_n$$

$$t_n(\rho) = L_n(\rho)\,t_n - M_n(\rho)\,u_n$$

Boundary representation:

$\rho \to \bar{\rho}$

$$0 = U_n^i(\bar{\rho})\,t_n - T_n^e(\bar{\rho})\,u_n$$

$$0 = L_n^e(\bar{\rho})\,t_n - M_n^i(\bar{\rho})\,u_n$$

$$U_n^i = 2\pi a\, J_n(k\bar{\rho})\, H_n^{(1)}(k\rho)$$

$$T_n^i = 2\pi k a\, J_n^{'}(k\bar{\rho})\, H_n^{(1)}(k\rho)$$

$$T_n^e = 2\pi k a\, J_n(k\rho)\, H_n^{(1)}{}'(k\bar{\rho})$$

$$L_n^i = 2\pi k a\, J_n(k\bar{\rho})\, H_n^{(1)}{}'(k\rho)$$

$$L_n^e = 2\pi k a\, J_n^{'}(k\rho)\, H_n^{(1)}(k\bar{\rho})$$

$$M_n^i = 2\pi k^2 a\, J_n^{'}(k\bar{\rho})\, H_n^{(1)}{}'(k\rho)$$

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Fictitious Eigenvalues by Indirect Method

- **Given** u_n

single layer density

$$\phi_n = \frac{u_n}{U_n(a)} \quad u(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{H_n^{(1)}(k\rho)\, J_n(ka)}{H_n^{(1)}(ka)\, J_n(ka)}\, u_n\, e^{-in\theta}$$

double layer density

$$\varphi_n = \frac{-u_n}{T_n(a)} \quad u(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{H_n^{(1)}(k\rho)\, J_n'(ka)}{H_n^{(1)}(ka)\, J_n'(ka)}\, u_n\, e^{-in\theta}$$

- **Given** t_n

single layer density

$$\phi_n = \frac{t_n}{L_n(a)} \quad t(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{H_n^{(1)}{}'(k\rho)\, J_n(ka)}{H_n^{(1)}{}'(ka)\, J_n(ka)}\, t_n\, e^{-in\theta}$$

double layer density

$$\varphi_n = \frac{-t_n}{M_n(a)} \quad t(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{H_n^{(1)}{}'(k\rho)\, J_n'(ka)}{H_n^{(1)}{}'(ka)\, J_n'(ka)}\, t_n\, e^{-in\theta}$$

Fictitious Eigenvalues by Direct Method (U,T Kernels)

- **Given** t_n

Using U, T integral equation :

unknown density $u_n = \dfrac{U_n(a)}{T_n^e(a)}\ t_n$

exact solution $$u(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{U_n(\rho)}{T_n^e(a)}\ t_n\ e^{-in\theta} = \sum_{n=-\infty}^{n=\infty} \frac{U_n(\rho)}{U_n(a)}\ u_n\ e^{-in\theta}$$

- **Given** u_n

Using U, T integral equation :

unknown density $t_n = \dfrac{T_n^e(a)}{U_n(a)}\ u_n$

exact solution $$u(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{U_n(\rho)}{U_n(a)}\ u_n\ e^{-in\theta} = \sum_{n=-\infty}^{n=\infty} \frac{H_n^{(1)}(k\rho)\ J_n(ka)}{H_n^{(1)}(ka)\ J_n(ka)}\ u_n\ e^{-in\theta}$$

Fictitious Eigenvalues by Direct Method (L,M Kernels)

- **Given** u_n

Using L, M integral equation :

unknown density $$t_n = \frac{M_n(a)}{L_n^e(a)} \, u_n$$

exact solution $$t(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{M_n(\rho)}{L_n^e(a)} \, u_n \, e^{-in\theta} = \sum_{n=-\infty}^{n=\infty} \frac{M_n(\rho)}{M_n(a)} \, t_n \, e^{-in\theta}$$

- **Given** t_n

Using L, M integral equation :

unknown density $$u_n = \frac{L_n^e(a)}{M_n(a)} \, t_n$$

exact solution $$t(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} \frac{M_n(\rho)}{M_n(a)} \, t_n \, e^{-in\theta} = \sum_{n=-\infty}^{n=\infty} \frac{H_n^{(1)}{}'(k\rho) \, J_n^{'}(ka)}{H_n^{(1)}{}'(ka) \, J_n^{'}(ka)} \, t_n \, e^{-in\theta}$$

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Factors Influence Fictitious Eigenvalues

- **Direct Method**

  U, T integral equation --- associated interior Dirichlet problem

  L, M integral equation --- associated interior Neumann problem

  Independent of boundary condition

- **Indirect Method**

  Single layer(U, L kernel) --- associated interior Dirichlet problem

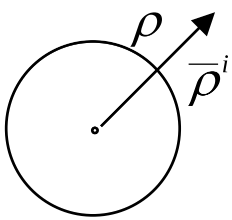
  Double layer(T, M kernel) --- associated interiorNeumann problem

  Location of singularity distribution

  Independent of boundary condition

Continuous Behavior of $U_n(\rho)$

(1) $\rho \to \overline{\rho}^{\,i}$

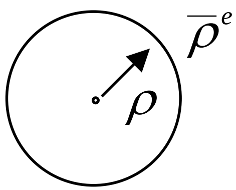


$$U^i(s,x) = \sum_{n=-\infty}^{n=\infty} J_n(k\overline{\rho})\, H_n^{(1)}(k\rho)\, e^{in(\overline{\theta}-\theta)}$$

$$U_n^i(\rho) = 2\pi a\, J_n(k\overline{\rho})\, H_n^{(1)}(k\rho)$$

$$= 2\pi a\, J_n(k\overline{\rho})\, [J_n(k\rho) + i\, Y_n(k\rho)]$$

(2) $\rho \to \overline{\rho}^{\,e}$



$$U^e(s,x) = \sum_{n=-\infty}^{n=\infty} J_n(k\rho)\, H_n^{(1)}(k\overline{\rho})\, e^{-in(\overline{\theta}-\theta)}$$

$$U_n^e(\rho) = 2\pi a\, J_n(k\rho)\, H_n^{(1)}(k\overline{\rho})$$

$$= 2\pi a\, J_n(k\rho)[J_n(k\overline{\rho}) + i\, Y_n(k\overline{\rho})]$$

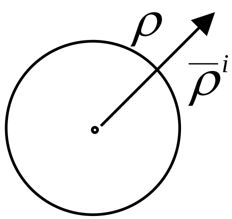
(3) real part: continuous

(4) imaginary part:continuous

$$\lim_{\rho\to a} U_n^i(\rho) = \lim_{\rho\to a} U_n^e(\rho)$$

Jump Behavior of $T_n(\rho)$

(1) $\rho \to \bar{\rho}^i$

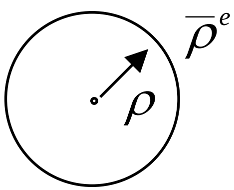


$$T^i(s,x) = \sum_{n=-\infty}^{n=\infty} k\, J_n'(k\bar{\rho})\, H_n^{(1)}(k\rho)\, e^{in(\bar{\theta}-\theta)}$$

$$T_n^i(\rho) = 2\pi ka\, J_n'(k\bar{\rho})\, H_n^{(1)}(k\rho)$$

$$= 2\pi ka\, J_n'(k\bar{\rho})\, [J_n(k\rho) + i\, Y_n(k\rho)]$$

(2) $\rho \to \bar{\rho}^e$



$$T^e(s,x) = \sum_{n=-\infty}^{n=\infty} k\, J_n(k\rho)\, H_n^{(1)}{}'(k\bar{\rho})\, e^{-in(\bar{\theta}-\theta)}$$

$$T_n^e(\rho) = 2\pi ka\, J_n(k\rho)\, H_n^{(1)}{}'(k\bar{\rho})$$

$$= 2\pi ka\, J_n(k\rho)[J_n'(k\bar{\rho}) + i\, Y_n'(k\bar{\rho})]$$

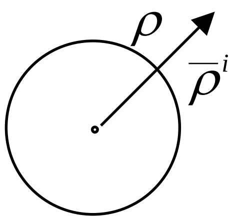
(3) real part: discontinuous -1 if $U(s,x) = \frac{i}{4} H_0^{(1)}(kr)$ **using Wronskian**

(4) imaginary part:continuous

$$W(J_n(ka), Y_n(ka)) = \frac{2}{\pi ka}$$

Jump Behavior of $L_n(\rho)$

(1) $\rho \to \bar{\rho}^i$

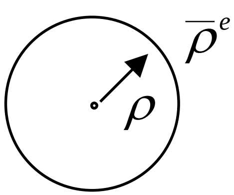


$$L^i(s,x) = \sum_{n=-\infty}^{n=\infty} k\, J_n(k\bar{\rho})\, H_n^{(1)}{}'(k\rho)\, e^{in(\bar{\theta}-\theta)}$$

$$L_n^i(\rho) = 2\pi ka\, J_n(k\bar{\rho})\, H_n^{(1)}{}'(k\rho)$$

$$= 2\pi ka\, J_n(k\bar{\rho})\, [J_n'(k\rho) + i\, Y_n'(k\rho)]$$

(2) $\rho \to \bar{\rho}^e$



$$L^e(s,x) = \sum_{n=-\infty}^{n=\infty} k\, J_n'(k\rho)\, H_n^{(1)}(k\bar{\rho})\, e^{-in(\bar{\theta}-\theta)}$$

$$L_n^e(\rho) = 2\pi ka\, J_n'(k\rho)\, H_n^{(1)}(k\bar{\rho})$$

$$= 2\pi ka\, J_n'(k\rho)[J_n(k\bar{\rho}) + i\, Y_n(k\bar{\rho})]$$

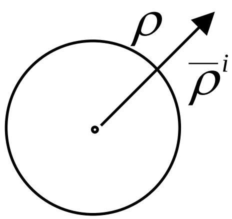
(3) real part: discontinuous 1 if $U(s,x) = \frac{i}{4} H_0^{(1)}(kr)$ **using Wronskian**

(4) imaginary part:continuous

$$W(J_n(ka), Y_n(ka)) = \frac{2}{\pi ka}$$

Continuous Behavior of $M_n(\rho)$

(1) $\rho \to \overline{\rho}^i$

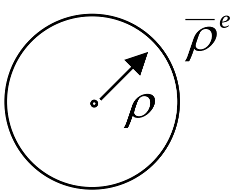


$$M^i(s,x) = \sum_{n=-\infty}^{n=\infty} k^2 J_n'(k\overline{\rho})\, H_n^{(1)}{}'(k\rho)\, e^{in(\overline{\theta}-\theta)}$$

$$M_n^i(\rho) = 2\pi k^2 a\, J_n'(k\overline{\rho})\, H_n^{(1)}{}'(k\rho)$$

$$= 2\pi k^2 a\, J_n'(k\overline{\rho})\, [J_n'(k\rho) + i\, Y_n'(k\rho)]$$

(2) $\rho \to \overline{\rho}^e$



$$M^e(s,x) = \sum_{n=-\infty}^{n=\infty} k^2 J_n'(k\rho)\, H_n^{(1)}{}'(k\overline{\rho})\, e^{-in(\overline{\theta}-\theta)}$$

$$M_n^e(\rho) = 2\pi k^2 a\, J_n'(k\rho)\, H_n^{(1)}{}'(k\overline{\rho})$$

$$= 2\pi k^2 a\, J_n'(k\rho)[J_n'(k\overline{\rho}) + i\, Y_n'(k\overline{\rho})]$$

(3) real part: continuous

(4) imaginary part:continuous

$$\lim_{\rho \to a} M_n^i(\rho) = \lim_{\rho \to a} M_n^e(\rho)$$

Relations of Internal Stiffness and External Stiffness

- **Internal stiffness** — **Physical natural frequency(U,T and L,M)**

$$\begin{bmatrix} U_n^e & -T_n^i \\ L_n^i & -M_n^e \end{bmatrix}\begin{Bmatrix} t_n \\ u_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

U,T method

$$\begin{cases} u = 0 & \Rightarrow J_n(ka) = 0 \\ t = 0 & \Rightarrow J_n^{'}(ka) = 0 \end{cases}$$

L,M method

$$\begin{cases} u = 0 & \Rightarrow J_n(ka) = 0 \\ t = 0 & \Rightarrow J_n^{'}(ka) = 0 \end{cases}$$

- **External stiffness** — **Fictitious eigen frequency(U,T and L,M):**

$$\begin{bmatrix} U_n^i & -T_n^e \\ L_n^e & -M_n^i \end{bmatrix}\begin{Bmatrix} t_n \\ u_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

U,T method

$$\begin{cases} u = \bar{u} & \Rightarrow J_n(ka) = 0 \\ t = \bar{t} & \Rightarrow J_n(ka) = 0 \end{cases}$$

L,M method

$$\begin{cases} u = \bar{u} & \Rightarrow J_n^{'}(ka) = 0 \\ t = \bar{t} & \Rightarrow J_n^{'}(ka) = 0 \end{cases}$$

- **Relations of stiffness**

$$\begin{bmatrix} U_n^e = U_n^i & -T_n^i = -L_n^e \\ L_n^i = T_n^e & -M_n^e = -M_n^i \end{bmatrix}\begin{Bmatrix} t_n \\ u_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} T_n^i - T_n^e = -1 \\ L_n^i - L_n^e = 1 \end{cases}$$

Continuous and Discontinuous Behavior of Dual Integral Equation

(real part)

$U(s,x)$

$T(s,x)$

$L(s,x)$

$M(s,x)$

Continuous and Discontinuous Behavior of Dual Integral Equation

(imaginary part)

$U(s,x)$

$T(s,x)$

$L(s,x)$

$M(s,x)$

Dependence of the Four Kernel Functions

- **U(s,x) and T(s,x) are linearly dependent on x since the Wronskian is zero**

$$W = \begin{vmatrix} U(s,x) & T(s,x) \\ \partial U(s,x)/\partial x & \partial T(s,x)/\partial x \end{vmatrix} = 0$$

- **L(s,x) and M(s,x) are linearly dependent on x since the Wronskian is zero**

$$W = \begin{vmatrix} L(s,x) & M(s,x) \\ \partial L(s,x)/\partial x & \partial M(s,x)/\partial x \end{vmatrix} = 0$$

Dependence of Dual Integral Equations

- **Dependence of primary field u(x) and secondary field t(x)**

  for U(s,x) and L(s,x)

$$W = \begin{vmatrix} u(x) & t(x) \\ \partial u(x)/\partial x & \partial t(x)/\partial x \end{vmatrix} = \begin{vmatrix} U(s,x) & L(s,x) \\ \partial U(s,x)/\partial x & \partial L(s,x)/\partial x \end{vmatrix}$$

  for T(s,x) and M(s,x)

$$W = \begin{vmatrix} u(x) & t(x) \\ \partial u(x)/\partial x & \partial t(x)/\partial x \end{vmatrix} = \begin{vmatrix} T(s,x) & M(s,x) \\ \partial T(s,x)/\partial x & \partial M(s,x)/\partial x \end{vmatrix}$$

- **Dependence(1-D):**

$$W = \begin{vmatrix} u(x) & t(x) \\ \partial u(x)/\partial x & \partial t(x)/\partial x \end{vmatrix} = \begin{vmatrix} C_n(kx) & kC_n^{'}(kx) \\ kC_n^{'}(kx) & k^2 C_n^{''}(kx) \end{vmatrix} = 0$$

  where $C_n(kx) = \frac{i}{k} e^{-ikx}$

- **Independence:**

$$W = \begin{vmatrix} u(x) & t(x) \\ \partial u(x)/\partial x & \partial t(x)/\partial x \end{vmatrix} = \begin{vmatrix} C_n(kx) & kC_n^{'}(kx) \\ kC_n^{'}(kx) & k^2 C_n^{''}(kx) \end{vmatrix} \neq 0$$

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Dependence of Undetermined Coefficients in Dual Integral Equations for Normal Boundary and Degenerate Boundary

- **The four constraints**

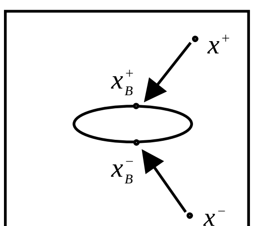
$$
\begin{array}{ll}
U^{+} & T^{+} \\
U^{-} & T^{-} \\
L^{+} & M^{+} \\
L^{-} & M^{-}
\end{array}
\begin{bmatrix}
C_n(kx^{+})R_n(kx_B^{+}) & C_n(kx^{+})R_n(kx_B^{-}) & kC_n^{'}(kx_B^{+})R_n(kx^{+}) & kC_n^{'}(kx_B^{-})R_n(kx^{+}) \\
C_n(kx^{-})R_n(kx_B^{+}) & C_n(kx^{-})R_n(kx_B^{-}) & kC_n^{'}(kx_B^{+})R_n(kx^{-}) & kC_n^{'}(kx_B^{-})R_n(kx^{-}) \\
kC_n(kx_B^{+})R_n^{'}(kx^{+}) & -kC_n(kx_B^{-})R_n^{'}(kx^{+}) & k^2C_n^{'}(kx^{+})R_n^{'}(kx_B^{+}) & -k^2C_n^{'}(kx^{+})R_n^{'}(kx_B^{-}) \\
-kC_n(kx_B^{+})R_n^{'}(kx^{-}) & kC_n(kx_B^{-})R_n^{'}(kx^{-}) & -k^2C_n^{'}(kx^{-})R_n^{'}(kx_B^{+}) & k^2C_n^{'}(kx^{-})R_n^{'}(kx_B^{-})
\end{bmatrix}
\begin{Bmatrix} t_n^{+} \\ t_n^{-} \\ u_n^{+} \\ u_n^{-} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

If $x^{+} \neq x^{-}$, (normal boundary) rank = 2

U^{+}, T^{+} and U^{-}, T^{-} are independent

L^{+}, M^{+} and L^{-}, M^{-} are independent

U, T and L, M equations are dependent

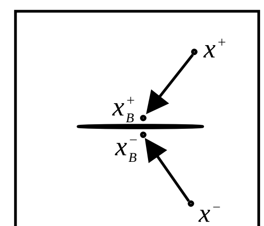


If $x^{+} = x^{-}$, (degenerate boundary) rank = 2

U^{+}, T^{+} and U^{-}, T^{-} are the same

L^{+}, M^{+} and L^{-}, M^{-} are different only in sign

U, T and L, M equations are independent



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Dependence of Undetermined Coefficients in Dual Integral Equations

- **Determinant is zero**

$$D = \lim_{x \to x_B} \begin{vmatrix} C_n(kx) R_n(kx_B) & kC_n^{'}(kx_B) R_n(kx) \\ kC_n(kx_B) R_n^{'}(kx) & k^2 C_n^{'}(kx) R_n^{'}(kx_B) \end{vmatrix} = 0$$

Dependent
for normal boundary !

- **Role of dual integral equations**

Fictitious eigenvalue

$$D = \lim_{x \to B} \begin{vmatrix} 0 & 0 \\ L^e & M^i \end{vmatrix} = 0, \text{ for } R_n(ka) = 0$$

$$D = \lim_{x \to B} \begin{vmatrix} U^i & T^e \\ 0 & 0 \end{vmatrix} = 0, \text{ for } R_n'(ka) = 0$$

Degenerate boundary

$$D = \lim_{x \to B} \begin{vmatrix} 0 & 0 \\ L^e & M^i \end{vmatrix} = 0, \text{ for subtraction}$$

$$D = \lim_{x \to B} \begin{vmatrix} U^i & T^e \\ 0 & 0 \end{vmatrix} = 0, \text{ for addition}$$

- **For static case,** $k \to 0$

Dependence of Dual Integral Equations

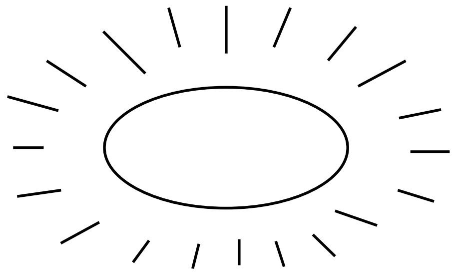
time harmonic $(\nabla^2 + k^2)u = 0 \rightarrow \nabla^2 u = 0$ **static**

ω: associated interior frequency

$\overline{\omega}$: excitation harmonic frequency

U, T kernel

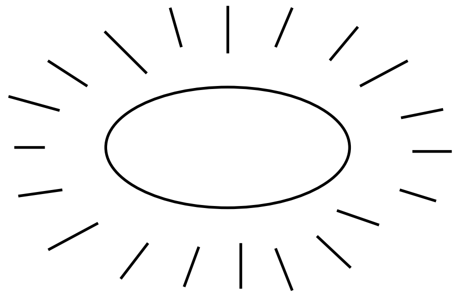
L, M kernel



$\overline{\omega} \rightarrow 0$

$k \rightarrow 0$

$kc = \overline{\omega}$



U, T kernel

L, M kernel

(either one needed)

Either one needed if

$R_n(ka) \neq 0$

$R_n'(ka) \neq 0$

$$D = \lim_{x \to B} \begin{vmatrix} 0 & 0 \\ L^e & M^i \end{vmatrix} = 0, \text{ for } R_n(ka) = 0$$

$$D = \lim_{x \to B} \begin{vmatrix} U^i & T^e \\ 0 & 0 \end{vmatrix} = 0, \text{ for } R_n'(ka) = 0$$

$$D = \lim_{x \to B} \begin{vmatrix} 0 & 0 \\ L^e & M^i \end{vmatrix} = 0, \text{ for subtraction}$$

$$D = \lim_{x \to B} \begin{vmatrix} U^i & T^e \\ 0 & 0 \end{vmatrix} = 0, \text{ for addition}$$

U, T kernel

L, M kernel

$\overline{\omega} = \omega$

$\overline{\omega} \neq \omega$

$\omega \rightarrow \infty$

degeneracy

$\overline{\omega} \rightarrow 0$

$k \rightarrow 0$

degeneracy

U, T kernel

L, M kernel

(both one needed)

(both one needed)

Comments on the Literature Work

- **Martin (1980)**

It is well known that both methods(potential method and Green's theorem) yield integral equations which have unique solution, except at the same discrete set of wave numbers(irregular values), corresponding to the eigenfrequencies of the interior Dirichlet Problem. The same methods can be modified to solve the exterior Dirichlet problem, and both yield equations of second kind which have unique solutions except at the frequencies of interior Neumann problem **(o)**

- **Shaw (1979)**

Exterior Dirichlet → interior Neumann eigenvalues
Exterior Neumann → interior Dirichlet eigenvalues **(?)**

- **Rizzo(1985, 1986), Hwang(1991)**

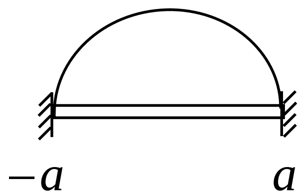
Fictitious eigenvalues are equal to eigenvalues of interior domain with reverse boundary conditions **(?)**

- **Huang(1989)**

Numerical experiments show that fictitious eigenvalue is indepedent on BC **(o)**

associated interior problem | **exterior problem**

Dirichlet B.C.

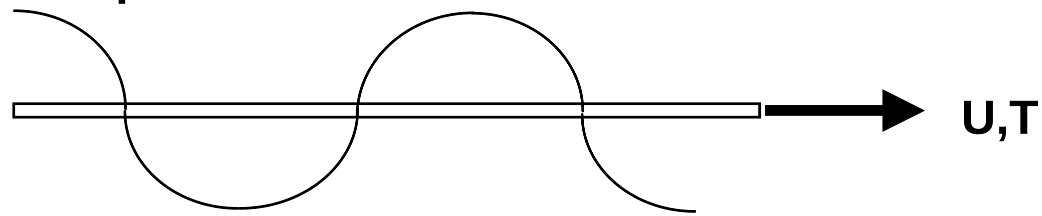


mode: $u(x) = \cos(kx)$

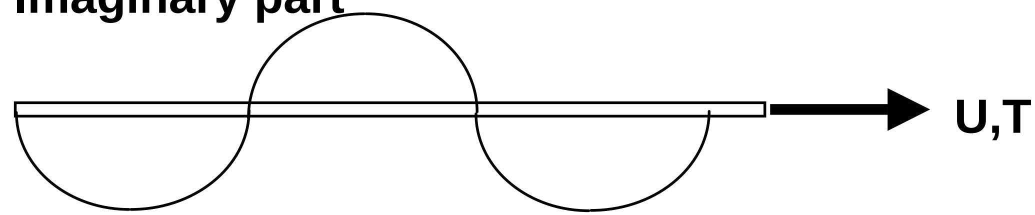
eigen equ.: $\cos(ka) = 0$

$k = 1/2, \quad a = \pi$

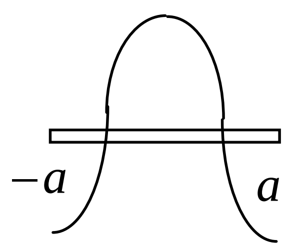
Real part



Imaginary part



Neumann B.C.

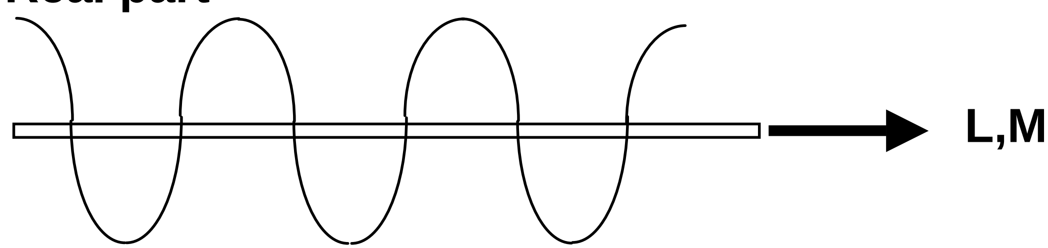


mode: $u(x) = \cos(kx)$

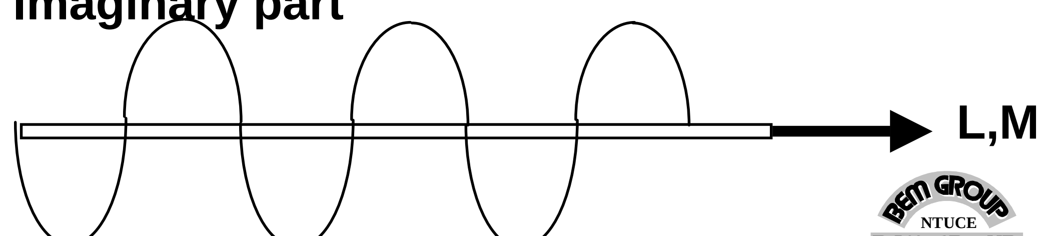
eigen equ.: $\sin(ka) = 0$

$k = 1, \quad a = \pi$

Real part



Imaginary part



Conclusions

- Fictitious eigenvalue depends on the integral representation of solution
- The numerical instability stems from 0/0
- The type of boundary condition can not change the position of fictitious eigenvalues once the representation is used
- The dependence of the two equations in dual representation model has been examined and the role of hypersingular equation has been discussed
- Three cases, 1-D, 2-D and 3-D, are demonstrated to see the beautiful structure of the mechanism for fictitious eigenvalues by direct and indirect BEM

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