Deverived the fundamental solution of $\frac{d^4 U(x,s)}{dx^4} = \delta(x-s),\ -\infty < x < \infty,$

by using Fourier transform, inverse Fourier transform and residue theorem.

(1).**Is** $U(x,s)$ **singular ?**

(2).**Is** $U(x,s)$ **symmetric?**

(3).**Is** $U(x,s)$ **degenreate form?**

(4) .**3 D Plot** $U(x,s)$ **and contou plot.**

Sol :

Fourier transform

$$\frac{d^4 U(x,s)}{dx^4} = \delta(x-s)$$

$$\longrightarrow (ik)^4\, \hat{U}(k,s) = e^{-iks}$$

$$\longrightarrow \hat{U}(k,s) = \frac{1}{k^4} e^{-iks}$$

Inverse Fourier transform

$$\longrightarrow U(x,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{k^4} e^{ik(x-s)}\, dk$$

let $k \longrightarrow z$

$$\longrightarrow \frac{1}{2\pi}\oint \frac{1}{z^4} e^{iz(x-s)}\, dk = 0$$

Taylor ' s series

$$e^{ik(x-s)} = 1 + (i(x-s))\, z + \frac{1}{2!}(i(x-s))^2 z^2 + \frac{1}{3!}(i(x-s))^3 z^3 + \cdots\cdots$$

(a), 上半面

$$\oint \frac{1}{z^4} e^{iz(x-s)}\, dk =$$

$$\int_{C_1} + \int_{C_2} + \int_{C_\rho}\left[\frac{1}{z^4} + \frac{1}{z^3}(i(x-s)) + \frac{1}{2z^2}(i(x-s))^2 + \frac{1}{6z}(i(x-s))^3\right] dk + \int_{C_R} = 0$$

$$\int_{C_\rho}\left[\frac{1}{z^4} + \frac{1}{z^3}(i(x-s)) + \frac{1}{2z^2}(i(x-s))^2 + \frac{1}{6z}(i(x-s))^3\right] dk$$

set $z = \rho e^{i\theta},\ dz = \rho i e^{i\theta} d\theta$

$$= \int_{\pi}^{0}\left[\frac{1}{\rho^4 e^{i4\theta}} + \frac{1}{\rho^3 e^{i3\theta}}(i(x-s)) + \frac{1}{2\rho^2 e^{i2\theta}}(i(x-s))^2 + \frac{1}{6\rho e^{i\theta}}(i(x-s))^3\right]\rho i e^{i\theta}\, d\theta$$

$$= \int_{\pi}^{0}\left[\frac{i}{\rho^3 e^{i3\theta}} + \frac{i}{\rho^2 e^{i2\theta}}(i(x-s)) + \frac{i}{2\rho e^{i\theta}}(i(x-s))^2 + \frac{i}{6}(i(x-s))^3\right] d\theta$$

$$= \frac{i}{\rho^3}\int_{\pi}^{0}(\cos[3\theta] - i\sin[3\theta])\, d\theta - \frac{(x-s)}{\rho^2}\int_{\pi}^{0}(\cos[2\theta] - i\sin[2\theta])\, d\theta -$$

$$\frac{i(x-s)}{2\rho}\int_{\pi}^{0}(\cos[\theta] - i\sin[\theta])\, d\theta + \frac{(x-s)^3}{6}\int_{\pi}^{0} 1\, d\theta$$

$$= \frac{-2}{\rho^3} + \frac{(x-s)}{\rho} + \frac{\pi(s-x)^3}{6}$$

$$\frac{1}{2\pi}\left[\int_{C_1+C_2} + \left(\frac{-2}{\rho^3} + \frac{(x-s)}{\rho}\right)\right] + \frac{(s-x)^3}{12} = 0$$

$$\Longrightarrow \frac{1}{2\pi}\left[\int_{C_1+C_2} + \left(\frac{-2}{\rho^3} + \frac{(x-s)}{\rho}\right)\right] = \frac{(x-s)^3}{12},\quad x > s$$

(b), 下半面

$$\oint \frac{1}{z^4} e^{iz(x-s)}\, dk =$$

$$\int_{C_1} + \int_{C_2} - \int_{C_\rho}\left[\frac{1}{z^4} + \frac{1}{z^3}(i(x-s)) + \frac{1}{2z^2}(i(x-s))^2 + \frac{1}{6z}(i(x-s))^3\right] dk - \int_{C_R} = 0$$

$$-\int_{C_\rho}\left[\frac{1}{z^4}+\frac{1}{z^3}(i(x-s))+\frac{1}{2z^2}(i(x-s))^2+\frac{1}{6z}(i(x-s))^3\right]dk$$

set $z=\rho e^{i\theta}$, $dz=\rho i e^{i\theta}d\theta$

=

$$-\int_\pi^0\left[\frac{1}{\rho^4 e^{i4\theta}}+\frac{1}{\rho^3 e^{i3\theta}}(i(x-s))+\frac{1}{2\rho^2 e^{i2\theta}}(i(x-s))^2+\frac{1}{6\rho e^{i\theta}}(i(x-s))^3\right]\rho i e^{i\theta}d\theta$$

$$=-\int_\pi^0\left[\frac{i}{\rho^3 e^{i3\theta}}+\frac{i}{\rho^2 e^{i2\theta}}(i(x-s))+\frac{i}{2\rho e^{i\theta}}(i(x-s))^2+\frac{i}{6}(i(x-s))^3\right]d\theta$$

$$=-\frac{i}{\rho^3}\int_\pi^0(\cos[3\theta]-i\sin[3\theta])d\theta+\frac{(x-s)}{\rho^2}\int_\pi^0(\cos[2\theta]-i\sin[2\theta])d\theta+$$

$$\frac{i(x-s)}{2\rho}\int_\pi^0(\cos[\theta]-i\sin[\theta])d\theta-\frac{(x-s)^3}{6}\int_\pi^0 1\,d\theta$$

$$=\frac{2}{\rho^3}-\frac{(x-s)}{\rho}-\frac{\pi(s-x)^3}{6}$$

$$\frac{1}{2\pi}\left[\int_{C_1+C_2}+\left(\frac{2}{\rho^3}-\frac{(x-s)}{\rho}\right)\right]-\frac{(s-x)^3}{12}=0$$

$$\Longrightarrow\frac{1}{2\pi}\left[\int_{C_1+C_2}+\left(\frac{-2}{\rho^3}+\frac{(x-s)}{\rho}\right)\right]=\frac{(s-x)^3}{12},\quad x<s$$

$$U(x,s)=\begin{cases}\frac{(x-s)^3}{12}, & x>s\\ \frac{(s-x)^3}{12}, & x<s\end{cases}=\begin{cases}-\frac{s^3}{12}+\frac{s^2x}{4}-\frac{sx^2}{4}+\frac{x^3}{12}, & x>s\\ \frac{s^3}{12}-\frac{s^2x}{4}+\frac{sx^2}{4}-\frac{x^3}{12}, & x<s\end{cases}$$

so $U(x,s)$ is regular.
$U(x,s)$ is symmetric.
$U(x,s)$ is degenerate form.

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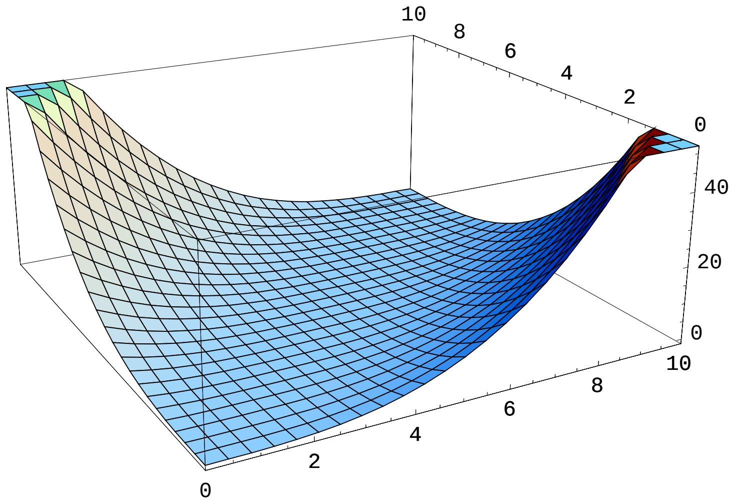
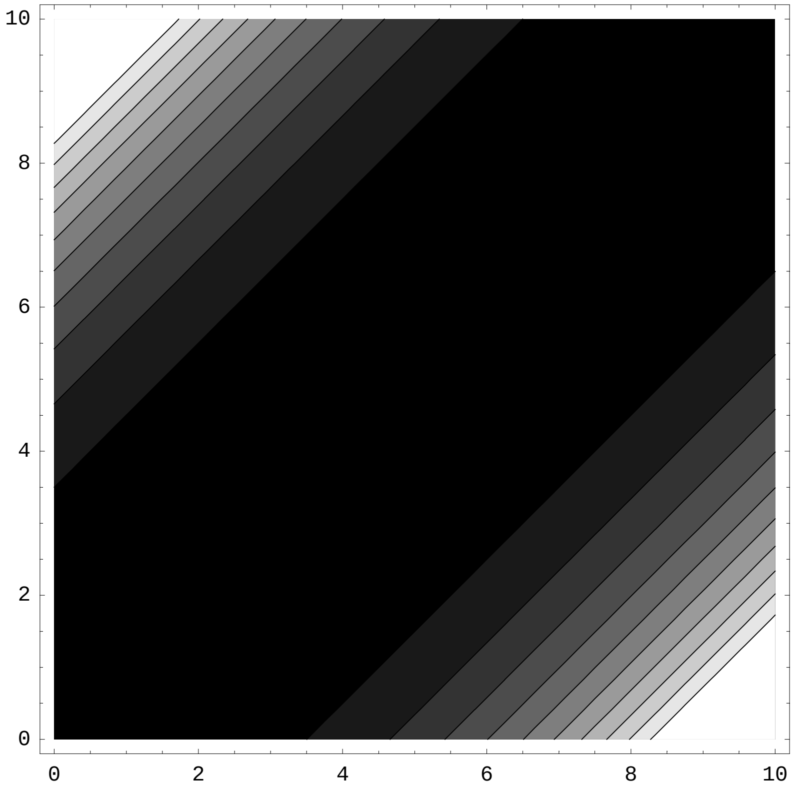
U1[x_, s_] := If[x > s, 1/12 (x - s)^3, 0]
U2[x_, s_] := If[x < s, 1/12 (s - x)^3, 0]
Plot3D[U1[x, s] + U2[x, s], {x, 0, 10}, {s, 0, 10}, ViewPoint → {-1.3, -2.4, 1}]
ContourPlot[U1[x, s] + U2[x, s], {x, 0, 10}, {s, 0, 10}]
```

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