$$
\begin{aligned}
&= \lim_{\epsilon \to 0}(\int_{b+\epsilon}^{c} -\phi(t)/(b-t)^2 \; dt - \phi(t)/(b-t) \mid_{b+\epsilon}) \\
&= \lim_{\epsilon \to 0}(-\phi(t)/(b-t) \mid_{b+\epsilon}^{c} + \int_{b+\epsilon}^{c} \phi'(t)/(b-t) \; dt + \phi(t)/(b-t) \mid_{b-\epsilon}) \\
&= \lim_{\epsilon \to 0} \int_{b+\epsilon}^{c} \phi'(t)/(b-t) \; dt - \phi(c)/(b-c) \qquad (1.224)
\end{aligned}
$$

當 $\epsilon \to 0$ 時，此積分可能不存在。

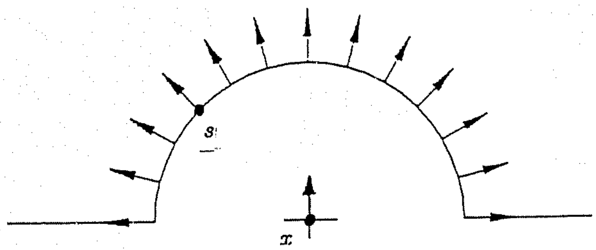
然而以下積分在 $\phi'(t)$ 於 $t = b$ 連續的情況下是存在的。

$$
\lim_{\epsilon \to 0}(\int_{a}^{b-\epsilon} + \int_{b+\epsilon}^{c})\phi'(t)/(b-t) \; dt = C.P.V. \int_{a}^{c} \phi'(t)/(b-t) \; dt
$$

1.10.4 主值與過奇異點積分技巧

主值的產生，可以由繞奇異點之半圓（二維問題）或半球面（三維問題）路徑的極限積分來解釋，以二維勢流場核函數說明如下:

範例: (a). 二維勢流場 $U(s,x)$



$U(s,x)$ 積分路徑示意圖（反時針）

因為

$$
U(s,x) = ln(r)
$$

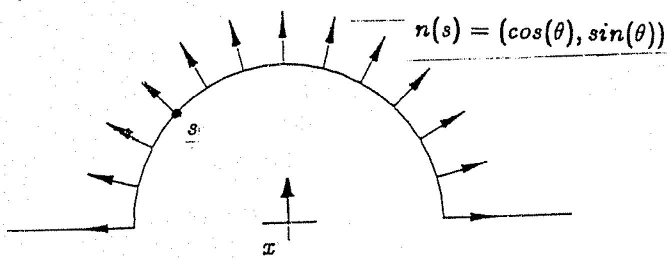
不失一般性，在局部行為，

$$
x = (0,0), \quad s = (\epsilon \cos\theta, \epsilon \sin\theta)
$$

$$
\begin{aligned}
\int U(s,x)t(s)dB(s) &= \int_{0}^{\pi} \ln\epsilon \, (\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta) \; \epsilon \; d\theta \\
&= \int_{0}^{\pi} \epsilon \; \ln\epsilon \, \frac{\partial u}{\partial x}\cos\theta \; d\theta + \int_{0}^{\pi} \epsilon \; \ln\epsilon \, \frac{\partial u}{\partial y}\sin\theta \; d\theta \\
&= -\epsilon \; \ln\epsilon \, \frac{\partial u}{\partial y}\cos\theta \mid_{0}^{\pi} = 2\epsilon \; \ln\epsilon \, \frac{\partial u}{\partial y}
\end{aligned}
$$

當 $\epsilon \to 0$ 時， $\epsilon\, ln\epsilon \to 0$ ，亦即繞奇異點之半圓路徑積分為 0 。故弱奇異核函數所產生之勢能是連續的。

範例: (b). 二維勢流場 $T(s,x)$ (強奇異)



$T(s,x)$ 積分路徑示意圖 (反時針)

其中

$$x=(0,0),\ s=(\epsilon\cos\theta,\epsilon\sin\theta),\ n(s)=(\cos\theta,\sin\theta),\ y_1=-\epsilon\cos\theta,\ y_2=-\epsilon\sin\theta$$

$u(s)$ 可在上半圓展開

$$\begin{aligned} u(s) &= u(x)+\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy \\ &= u(x)+\frac{\partial u}{\partial x}\epsilon\cos\theta+\frac{\partial u}{\partial y}\epsilon\sin\theta \end{aligned}$$

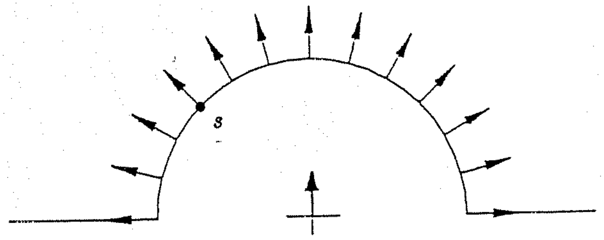
因為

$$T(s,x)=-y_in_i/r^2=\frac{\epsilon(cos^2(\theta)+sin^2(\theta))}{r^2}=\frac{1}{\epsilon}$$

$$\int \frac{-y_1n_1-y_2n_2}{r^2}\ [\ u(x)+\frac{\partial u}{\partial x}\epsilon\cos\theta+\frac{\partial u}{\partial y}\epsilon\sin\theta)\ ]\ \epsilon\ d\theta=\pi u$$

上式，上半圓之路徑積分值為 πu (up contour) ，下半圓之路徑積分值為 $-\pi u$ (down contour) ，表明了橫越邊界時跳躍 $2\pi u$ 。故強奇異核函數所產生之勢能為跳躍函數。

範例: (c). 二維勢流場 $L(s,x)$,(強奇異)



$L(s,x)$ 積分路徑示意圖 (反時針)

因為　$L(s,x) = y_i \bar{n}_i / r^2$，同理，可得

$$x = (0,0)$$
$$s = (\epsilon\cos\theta, \epsilon\sin\theta)$$
$$n(x) = (0,1)$$
$$t(s) = \frac{\partial u}{\partial x} n_1 + \frac{\partial u}{\partial y} n_2$$

其中 s 　係延著上半圓路徑積分

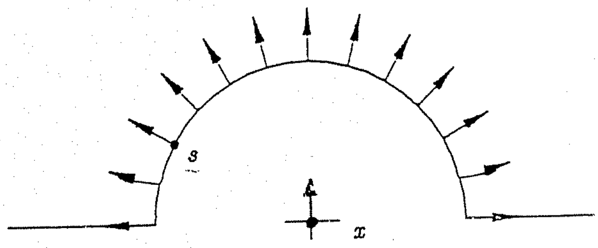
$$y_1 = -\epsilon\cos\theta\ ,$$
$$y_2 = -\epsilon\sin\theta$$
$$L(s,x) = \frac{-\epsilon sin(\theta)}{\epsilon^2}$$

$$\int L(s,x)\ t(s)\ dB(s)$$
$$= \int_0^\pi \frac{-\epsilon\sin\theta}{\epsilon^2} (\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta)\ \epsilon\ d\theta - \int_0^\pi \frac{\partial u}{\partial y}\sin^2\theta\ \epsilon\ d\theta$$
$$= -\frac{\pi}{2}\frac{\partial u}{\partial y} = -\frac{\pi}{2}t$$

上式，上半圓之路徑積分值為 $-\pi t/2$ (up contour) ，下半圓之路徑積分值為 $\pi t/2$ (down contour) ，表明了橫越邊界時跳躍 - πt 。故強奇異核函數所產生之勢能為跳躍函數。值得一提的是，針對相同的密度函數而言，單層勢能之法向微分場與雙層勢能場之和是連續的，列式如下:

$\int\{T(s,x) + L(s,x)\}\mu(s)ds$ 為連續。

範例: (d). 二維勢流場 $M(s,x)$ (超強奇異核函數)



$M(s,x)$ 積分路徑示意圖 (反時針)

因為

$$M(s,x) = 2y_i y_j n_i \bar{n}_j / r^4 - n_i \bar{n}_i / r^2$$

$$(4) \quad \int M(s,x)\, u(s)\, dB(s)$$

$$\begin{aligned} x =& \ \ (0,0),\ \ s = (\epsilon\cos\theta, \epsilon\sin\theta) \\ n(x) =& \ \ (0,1),\ \ n(s) = (\cos\theta, \sin\theta) \\ y_1 =& \ \ -\epsilon\cos\theta,\ \ y_2 = -\epsilon\sin\theta \end{aligned}$$

$$u(s) = u(x) + (\frac{\partial u}{\partial x}\epsilon\cos\theta + \frac{\partial u}{\partial y}\epsilon\sin\theta)$$

$$\begin{aligned} & \int M(s,x)\, u(s)\, dB(s) \\ &= \int (\frac{2y_i y_j n_i \bar{n}_j}{r^4} - \frac{n_i \bar{n}_i}{r^2})\ [\ u(x) + (\frac{\partial u}{\partial x}\epsilon\cos\theta + \frac{\partial u}{\partial y}\epsilon\sin\theta)\ ]\ dB \\ &= \int_0^\pi (\frac{2\epsilon^2\sin\theta}{\epsilon^4} - \frac{\sin\theta}{\epsilon^2})\ [\ u(x) + \epsilon(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta)\ ]\ \epsilon d\theta \\ &= \int_0^\pi (\frac{\sin\theta}{\epsilon^2})\ [\ u(x) + \epsilon(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta)\ ]\ \epsilon d\theta \\ &= \frac{\cos\theta}{\epsilon}|_0^\pi + \frac{\partial u}{\partial y}(\frac{1}{2})|_0^\pi \\ &= \frac{-2}{\epsilon} + \frac{\pi}{2}t \end{aligned}$$

上式，繞上半圓之積分值等於 $\frac{-2}{\epsilon} + \frac{\pi}{2}t$ ，繞下半圓之積分值等於 $\frac{-2}{\epsilon} - \frac{\pi}{2}t$ 。此無限值 $2/\epsilon$ 和H.P.V. 定義之 $-2/\epsilon$ 恰可消掉，而求出有限值 $\pi t/2$ ，故超強奇異核函